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*

'I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and an ornament thereunto.'
BACON

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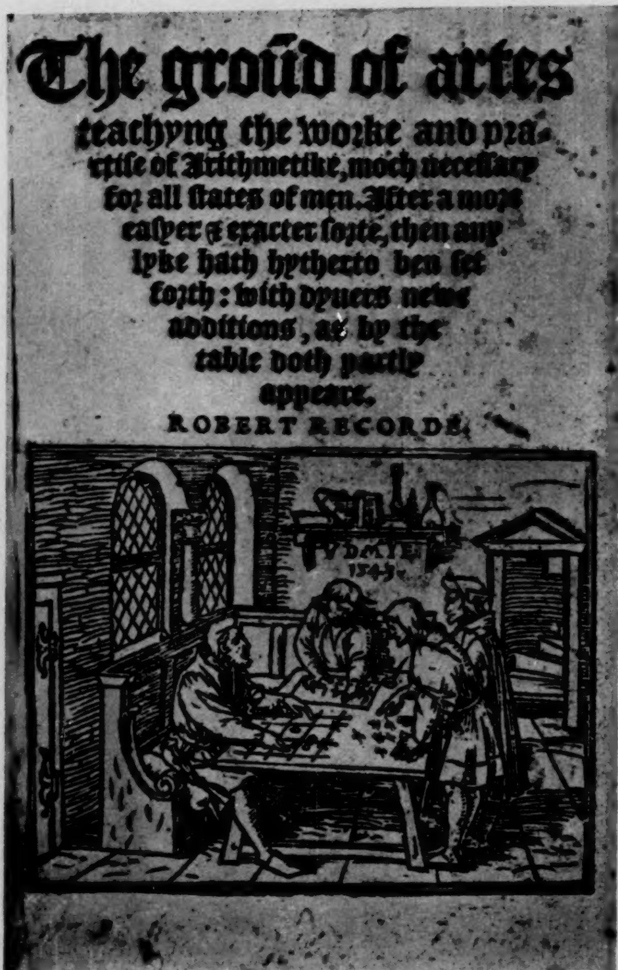

The groũd of artes teachyng the worke and practise of Arithmetike, moch necessary for all states of men. After a more easyer & exacter sorte, then any lyke hath hytherto ben set forth: with dyuers newe additions, as by the table doth partly appeare.

ROBERT RECORDE.

ROBERT RECORDE. 1510-1558

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"Think, David, think"

BY WILLIAM H. GLENN

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Because of the many papers that teachers must grade, it sometimes develops that a problem solved incorrectly is checked quickly by the teacher as wrong, and returned to the pupil without the teacher's having had a sufficient opportunity to know what the pupil was thinking when he did the problem. Some errors may be classified as pure carelessness, but in other cases there may be some basic fundamentals that the pupil has not grasped, and these are important to clarify.

The purpose of this paper is to show by two examples why a teacher needed to do more than put a mark beside an incorrect problem.

On one paper turned in by David, he showed the following work:

$$\begin{array}{r} 45 \\ \times\ 5 \\ \hline 305 \end{array}$$

Next to the solution, the teacher wrote a large *X* with the words, "Think, David, think!"

David was thinking, but the teacher interpreted the answer to be carelessness on his part. What the teacher needed to do was write on the paper or say in class, "See me about this problem, David." Then, when David could be asked to say aloud how he had arrived at his answer, David explained, "Well, 5 times 5 is 25. There is a 2 to carry; 4 and 2 are 6, and 5 times 6 is 30. So the answer is 305."

Of course David was thinking. He was using a principle about carrying that he had learned in addition problems, and he was modifying it to fit the situation when he multiplied. The teacher's

next step is clear. He can now use a pocket chart or some other meaningful teaching device to show the differences in the two processes and the why of each.

Another problem from David's paper was:

$$\begin{array}{r} 46 \\ -12 \\ \hline 38 \end{array}$$

And again the teacher wrote, "Think, David, think!" But again David was thinking. He said, when asked to do it orally, "There is a minus sign in front of the 1 so I subtracted it from 4, giving 3. There is no minus sign in front of the 2 so I added it to 6, making the answer 8."

In both of these examples, the symbolism of our place value system is involved. In the first case, 45 times 5, it is necessary to understand that 45 means 4 tens and 5 ones, or $(40 + 5)$. To multiply by 5 means to multiply *both parts* by 5, resulting in 20 tens and 25 ones, or $(200 + 25)$, which finally becomes 225. I am sure David could follow through in the same way if he were asked to multiply by 5 any one of the following: 4 dimes and 5 cents, or 4 pounds 5 ounces, or 4 feet 5 inches. In each case, 5 multiplies both the 4 and the 5, and the results are to be added to give the final answer.

In the second problem, the -12, by convention, means that the whole number 12 is to be subtracted. Even though 12 is one number, it is made up of two parts, 1 ten and 2 ones, and if 12 is to be subtracted, both of its parts must necessarily be subtracted.

These two problems are examples of the use of the distributive law; and although David does not need to know this law by name, as an elementary school student, he certainly will be faced with its use repeatedly when he reaches the level of a beginning algebra student. In fact, some of the most common errors of algebra students arise through the improper use of the distributive law. The first example,

$$5 \times 45 = 5(40 + 5) = 5 \times 40 + 20 + 5,$$

was done incorrectly by changing the order of operations to the following:

$$5(40 + 20) + 5$$

This has its counterpart in algebra as

$$5a + b + c$$

and when students are asked to substitute values in this expression,

some may do so by adding the values of a and b before multiplying by 5:

$$5(a + b) + c$$

The other problem,

$$(40 + 6) - (10 + 2),$$

has a counterpart in algebra of the type,

$$(4a + 3b) - (a + b),$$

which is solved incorrectly by some by writing this as

$$4a + 3b - a + b = 3a + 4b,$$

thus failing to subtract *both* the a and the b.

So don't assume that David isn't thinking when a wrong answer appears on his paper. Indeed he *is* thinking; but you, his teacher, must set his thinking straight.

W. H. G.

APPLICATIONS OF DIGITAL COMPUTERS IN MATHEMATICS*

By C. B. Haselgrove

Digital Computers can be thought to have 3 parts; *the memory*, which stores numbers and instructions, *the arithmetic unit*, which does the arithmetic, and *the control*, which selects, interprets, and acts upon the instructions. Numbers are fed into the machine by some means such as paper tape, which is prepared by a device like a typewriter. The instructions are put on the tape in a coded form, and may look like something like this:

$$A + B \rightarrow C$$

which would mean take the number in storage register A, add the number in storage register B and put the answer in storage register C. Functions like e^x can be worked out by their power series.

Among the first machines in Britain, we have the Mark I machine at Manchester, and the EDSAC I at Cambridge. These machines obey about 500–1,000 instructions per second, and could work out e^x in 1/10th of a second.

Later machines such as Mercury, EDSAC II and many American machines like the I.B.M. 704, are about 60 times as fast and are

* Abstract of a lecture given to the Mathematical Association, April 1958.

more versatile. Machines can be used to study a large variety of problems: design of aeroplanes, bridges, atomic piles, the calculation of income tax for a firm and big accounting problems. In physics they can be used to study atomic and molecular structure, structure of crystals, hydrodynamics and stellar evolution.

After the Inaugural Meeting of this Conference, I used the Mercury here to find out the structure of stars of the main sequence; it took about 7 hours to get the results. The EDSAC I had taken one night a week for six months to get comparable results.

In pure mathematics, machines can be used to obtain a lot of information which will enable mathematicians to make a good guess at the solution of a problem; for example Mordell & Woolett studied the solutions of the Diophantine equation $x^3 + y^3 + z^3 = k$ for fixed k and from solutions that they obtained Mordell was led to prove that there exists an infinity of solutions if k is a perfect cube.

Let $\lambda(n) = +1$ if n has an even number of prime factors and
$= -1$ if n has an odd number of prime factors

and let $L(x) = \sum_{n<x} \lambda(n)$. Pólya noticed that $L(x)$ was negative or zero for $2 \leqslant x < 1{,}500$ and conjectured that $L(x) \leqslant 0$ for all x.

I believe I have now disproved this conjecture and that $L(x)$ changes sign near $x = 1{\cdot}85 \times 10^{361}$. Even a computer which could do a million operations a second (and such computers are now being planned) could not do this directly in the life-time of the earth. It was done by computing the zeros of the Riemann zeta function $\zeta(s) = \Sigma n^{-s}$.

Riemann conjectured that all of the non-trivial zeros of the function $\zeta(s)$ lie on the line $R(s) = \frac{1}{2}$. Lehmer has proved, using an American computer, that the first 25,000 zeros of this function lie on this line. I have worked out the first 1500 zeros accurately, and it was those which I used to disprove Pólya's conjecture.

Another interesting problem for which computers can be used: can one find fractions p/r, q/r which approximate simultaneously to two irrational numbers θ and ϕ? The EDSAC has been used to find the most difficult pair of numbers θ and ϕ. Machines can also be programmed so that they handle mathematical expressions and combine them according to the laws of algebra, rather than arithmetic. They also can be programmed to perform the operations of logic and there is always hope that in the future a faster machine will be able to *prove* mathematical theorems by the process of deduction, rather than to "suggest" theorems.

C. B. H.

The University of Manchester

PARTIAL FRACTIONS: RATIONALIZING THE DENOMINATOR

By E. H. Neville

1. Since the fraction associated with $y+1$ in the decomposition of $2/(y+1)(y-1)$ is $-1/(y+1)$, the fraction associated with x^2+1 in the decomposition of $2/(x^2+1)(x^2-1)$ is $-1/(x^2+1)$, and therefore the fraction associated with x^2+1 in the decomposition of $2/(x^2+1)(x-1)$ is $-(x+1)/(x^2+1)$.

In this application of the threadbare device of rationalizing the denominator, which is ignored in the otherwise admirable sections on partial fractions in the recent Algebra Report, the numerator corresponding to a linear factor is not found unless it is wanted. To take the example $P_1/Q_1 \equiv x^3/(x^2+1)(2x-1)(x-1)$ from p. 41 of the Report,

$$\begin{aligned} P_1/Q_1 &= x^3(2x+1)(x+1)/(x^2+1)(4x^2-1)(x^2-1) \\ &= \{x(2x^4+x^2)+3x^4\}/(x^2+1)(4x^2-1)(x^2-1), \end{aligned}$$

and because the fractions associated with $y+1$ in the decomposition of the two fractions

$$(2y^2+y)/(y+1)(4y-1)(y-1),\ 3y^2/(y+1)(4y-1)(y-1)$$

are $1/10(y+1)$ and $3/10(y+1)$, the fraction associated with x^2+1 in the decomposition of P_1/Q_1 is $(x+3)/10(x^2+1)$.

In practice the work can be abbreviated, for -1 can be substituted for y at any stage, and therefore the symbol y is superfluous; instead of replacing x^2 by y and y by -1 we replace x^2 by -1. The numerator associated with x^2+1 in the decomposition of P_1/Q_1

$$\begin{aligned} &= [x^3/(2x^2-3x+1)]_{x^2=-1} \\ &= [x/(3x+1)]_{x^2=-1} \\ &= [(3x^2-x)/(9x^2-1)]_{x^2=-1} \\ &= (x+3)/10. \end{aligned}$$

We have nothing to gain by knowing the linear factors of $Q_1/(x^2+1)$; on the contrary, we must combine the factors if we are to reduce the denominator $Q_1/(x^2+1)$ to a linear form before rationalizing it.

2. The process of rationalization is specially to be recommended if the denominator has a quadratic factor with rational coefficients and irrational or complex roots. Take for example

$$P_2/Q_2 = 2(x^3+1)/(x^2+1)(x^2+2x-7).$$

The numerator associated with x^2+1

$$\begin{aligned}
&= [(-x+1)/(x-4)]_{x^2=-1} \\
&= [(-x^2-3x+4)/(x^2-16)]_{x^2=-1} \\
&= (3x-5)/17.
\end{aligned}$$

It is manifestly absurd to make the determination of this numerator consequent on the evaluation of $2(x^3+1)/(x+2\sqrt{2}+1)(x^2+1)$ or of $(x^3+1)/(2x^3+3x^2-6x+1)$ for $x=2\sqrt{2}-1$.

3. We can go further than this. If the fractions associated with the linear factors of x^2+2x-7 are wanted, it is well worth while to isolate the fraction associated with this quadratic factor before introducing surds. The numerator required

$$\begin{aligned}
&= [2(x^3+1)/(x^2+1)]_{x^2=-2x+7} \\
&= [(-2x^2+7x+1)/(-x+4)]_{x^2=-2x+7} \\
&= [(11x-13)(x+6)/(-x+4)(x+6)]_{x^2=-2x+7} \\
&= [(11x^2+53x-78)/(-x^2-2x+24)]_{x^2=-2x+7} \\
&= (31x-1)/17.
\end{aligned}$$

The check,

$$(3x-5)(x^2+2x-7)+(31x-1)(x^2+1)=34(x^3+1),$$

is simplicity itself, and now

$$\frac{2(31x-1)}{x^2+2x-7}=\frac{31-8\sqrt{2}}{x-2\sqrt{2}+1}+\frac{31+8\sqrt{2}}{x+2\sqrt{2}+1}.$$

Fundamentally the logic is the same for the factor x^2+2x-7 as for the factor x^2+1. The substitution $x+1=z$ followed by $y=z^2$ gives the second problem the form of the first, and since the factor $y-8$ is derived from x^2+2x-7 by the double substitution, we can lower the degrees of the polynomials involved by repeated use of the condition $x^2=-2x+7$ instead of by the explicit use of the substitution $z^2=8$. When the denominator of a function is reduced to the linear form $fx+g$, the introduction of the factor $f(x+2)-g$ converts the denominator into the constant $7f^2+2fg-g^2$.

4. One advantage of the approach by rationalization is that the form of the fractions to be associated with a repeated quadratic factor is clear from the outset. Let the fraction to be decomposed be P/X^nQ, where $X=(x-b)^2-a$ and P, Q are polynomials in x. The effect of the double substitution $x-b=z$, $z^2=y$ followed by rationalization of the denominator is to convert the fraction to the form $\{zp_1(y)+p_2(y)\}(y-a)^nq(y)$, that is, to the form $\{xf(y)+g(y)\}/(y-a)^nq(y)$, and since the fractions associated with $y-a$ in the decomposition of $f(y)/(y-a)^nq(y)$ and $g(y)/(y-a)^nq(y)$

compose two sums $\Sigma f_r/(y-a)^r$, $\Sigma g_r/(y-a)^r$, the fractions associated with X in the decomposition of P/X^nQ compose a sum $\Sigma(f_rx+g_r)/X^r$. Again we have no need of explicit subsidiary variables. The first step towards determining f_r and g_r from $f(y)$, $g(y)$, and $q(y)$ is to introduce $y-a$ as a new variable, and this variable is nothing but X itself. That is to say, $P(x)/Q(x)$ is necessarily expressible in the form $\{xR(X)+S(X)\}/T(X)$, and the set of fractions associated with X in P/X^nQ is $\Sigma(f_rx+g_r)/X^r$ where $\Sigma f_r/X^r$, $\Sigma g_r/X^r$ are the sets associated with X in R/X^nT and S/X^nT.

5. There are two routes from theory to practice, both of them much shorter than the Report would lead us to suppose. If $X=x^2-cx-d$, either repeated division by x^2-cx-d with remainders linear in x, or repeated substitution of $X+cx+d$ for x^2, can be used to bring $P(x)$, $Q(x)$ to the forms xM_P+N_P, xM_Q+N_Q, where M_P, N_P, M_Q, N_Q are polynomials in X. To rationalize the denominator we introduce the factor $(x-c)M_Q-N_Q$ and we have $P/Q=(xR+S)/T$, where

$$T=(X+d)M_Q^2-N_Q(cM_Q+N_Q),$$

$$R=N_PM_Q-M_PN_Q,\quad S=(X+d)M_PM_Q-N_P(cM_Q+N_Q).$$

All that remains, whatever the value of n, is to find the fractions associated with X in R/X^nT and S/X^nT.

Since the required fractions depend only on the terms in R, S, T of degree lower than n, we can ignore powers from X^n upwards throughout the work of transforming P/Q. This is precisely what we have done in §§2-3 for the case $n=1$, for each of the quadratic factors of Q_2. Instead of deriving from the relations $X_1=x^2+1$, $X_2=x^2+2x-7$ the identities

$$\frac{x^3+1}{X_1X_2}=\frac{x(X_1^2-9X_1+6)-(2X_1^2-5X_1+10)}{X_1(X_1^2-20X_1+68)}$$

$$=\frac{x(X_2^2+15X_2+62)-(X_2+2)}{X_2(X_2^2+16X_2+68)}$$

we have there shown only that if $X_1=0$, then $(x^3+1)/X_2=(3x-5)/34$, and that if $X_2=0$, then $(x^3+1)/X_1=(31x-1)/34$. If the fraction to be decomposed is $2(x^3+1)/X_1X_2^2$, to find the fractions associated with X_2 we retain the terms linear in X_2 in the numerator and denominator of $2(x^3+1)/X_1$; the fractions required are those associated with X_2 in

$$\{x(15X_2+62)-(X_2+2)\}/X_2^2(8X_2+34).$$

If

$$X_2=17Y,$$

then

$$34 + 8X_2 = 34(1 + 4Y),$$

and since the terms linear in Y in

$$\{x(62 + 255Y) - (2 + 17Y)\}/(1 + 4Y)$$

are

$$x(62 + 7Y) - (2 + 9Y),$$

the group of fractions associated with X_2 in $2(x^3 + 1)/X_1X_2^2$ is

$$(31x - 1)/17X_2^2 + (7x - 9)/578X_2.$$

6. The simplicity of the operation in §§2-3 is a recommendation to an alternative process in which the terms of the sum $\Sigma\,(f_rx + g_r)/X^n$ are peeled off from P/X^nQ one by one. The numerator for X in P/XQ is the linear function $fx + g$ such that $P - (fx + g)Q$ has X for a factor, and is therefore the numerator $f_nx + g_n$ for X^n in P/X^nQ. If $P - (f_nx + g_n)Q = XP_1$, then $P/X^nQ = (f_nx + g_n)/X^n + P_1/X^{n-1}Q$; the numerator for X^{n-1} in P_1/X^{n-1} is the numerator for X in P_1/XQ, and the operation can be repeated.

To find in this way the fractions associated with X_2 in

$$2(x^3 + 1)/X_1X_2^2,$$

we take the numerator $(31x - 1)/17$ for X_2^2 from §3. Then

$$34(x^3 + 1) - (31x - 1)(x^2 + 1) = (3x - 5)(x^2 + 2x - 7),$$

that is,

$$17P/X_1X_2^2 = (31x - 1)/X_2^2 + (3x - 5)/X_1X_2,$$

and since the numerator of X_2 in $(3x - 5)/X_1X_2$

$$\begin{aligned} &= [(3x - 5)/(-2x + 8)]_{x^2=-2x+7} \\ &= [(3x - 5)(x + 6)/34]_{x^2=-2x+7} \\ &= (7x - 9)/34, \end{aligned}$$

the pair of fractions associated with X_2 is

$$(31x - 1)/17X_2^2 + (7x - 9)/578X_2,$$

as found otherwise in § 5. The identity

$$34(3x - 5) - (7x - 9)(x^2 + 1) = -(7x - 23)(x^2 + 2x - 7)$$

verifies the calculation and provides the complete decomposition

$$2(x^3 + 1)/X_1X_2^2$$
$$= (31x - 1)/17X_2^2 + (7x - 9)/578X_2 - (7x - 23)/578X_1.$$

Found directly, the numerator of X_1 in $2(x^3 + 1)/X_1X_2^2$

$$= [2(x^3 + 1)/(x^2 + 2x - 7)^2]_{x^2=-1}$$

$$= [(-x + 1)/2(x^2 - 8x + 16)]_{x^2=-1}$$
$$= [(x - 1)/2(8x - 15)]_{x^2=-1}$$
$$= [(x - 1)(8x + 15)/2(64x^2 - 225)]_{x^2=-1}$$
$$= -(7x - 23)/578$$

as expected.

7. The Report refers to the substitution of complex values of x as "a useful short cut for quadratic factors". It must be understood that here too there is no question of repeated factors. It is only if $r = 1$ that the sum of two conjugate fractions $c_r/(x - a)^r$, $\bar{c}_r/(x - \bar{a})^r$ is a fraction $(p_r x + q_r)/\{(x - a)(x - \bar{a})\}^r$ with a linear numerator. In other words, if $n > 1$ there is no correspondence between a standard fraction associated with $(x - a)(x - \bar{a})$ and a pair of fractions associated with $x - a$ and $x - \bar{a}$, and if the decomposition is required in one form nothing is to be gained by effecting it first in the other form. If $n = 1$, the one fraction is the sum of the other two, and in this case the arithmetic is virtually the same in the two processes, for to evaluate a polynomial $P(x)$ when $x = b + ic$, it is far quicker first to reduce $P(x)$ to a linear form by repeated substitution of $2bx - (b^2 + c^2)$ for x^2 than to substitute $(b + ic)^r$ for x^r in the individual terms of $P(x)$.

The cynical advice in the Report, that a candidate encountering in an examination a problem involving repeated irreducible factors should turn to another question, is perhaps a slur on the examiner. If the mark-scheme offers anything approaching a reasonable return for time spent by the misguided candidate who uses undetermined coefficients—"We mustn't penalize the unfortunate boy for having been badly taught"—the candidate who uses an intelligent method will reap a very handsome reward in time saved and is likely to qualify for a bonus on his attempt even if his arithmetic goes wrong.

8. In performing a division, or a series of divisions, by a polynomial X of any degree in a variable x, it is almost always worth while to preface the operation of division by an explicit change of variable and of divisor to secure unity for the leading coefficient. If $X = ax^2 + bx + c$, and if the work is in descending powers of x, we write $y = ax$, $Y = aX = y^2 + by + ac$, and if the dividend P is of degree n, we obtain the identity $a^n P = QY + (py + q)$ without fractional coefficients; if the work is in ascending powers, the substitution is $x = cy$, and the divisor Y is given by

$$X = cY = c(1 + by + acy^2);$$

restitution of x and X can be made at any time. An example of this avoidance of fractions is given at the end of §5 above, and I should myself set out the division on p. 52 of the Report as the division

of $4 + 28z + 49z^2$ by $1 + 5z + 7z^2$; the identity found is

$$P = \frac{1}{7}(4 + 8z - 19z^2)Q + z^3R,$$

and the mounting powers of 7 come in only as denominators at the end. It is in accordance with this principle that the quadratic factor with which partial fractions are to be associated has been assumed to have the form $x^2 - cx - d$. The fractions associated with $3x^2 + 2x - 7$ in the fraction

$$(x^3 + 1)/(3x^2 + 2x - 7)^2(2x^2 + x + 1)$$

are to be derived from those associated with $y^2 + 2y - 21$ in the fraction

$$3(y^3 + 27)/(y^2 + 2y - 21)^2(2y^2 + 3y + 9),$$

and the fraction associated with $2x^2 + x + 1$ in the same fraction, if found independently, is to be derived from the fraction associated with $y^2 + y + 2$ in the fraction

$$4(y^3 + 8)/(y^2 + y + 2)(3y^2 + 4y - 28).$$

If $y^2 + 2y - 21 = Y$, then $2y^2 + 3y + 9 = -y + (2Y + 51)$. The rationalized denominator

$$\{-y + (2Y + 51)\}\{(y + 2) + (2Y + 51)\}$$

is

$$2682 + 207Y + 4Y^2,$$

and instead of dividing by the linear function $2682 + 207Y$ we write $Y = 298Z$ and divide by $1 + 23Z$. Alternatively,

$$[(y^3 + 27)/(2y^2 + 3y + 9)]_{y^2=-2y+21} = (70y - 15)/149,$$

$$149(y^3 + 27) - (70y - 15)(2y^2 + 3y + 9)$$
$$= 9(y - 22)(y^2 + 2y - 21),$$

$$[9(y - 22)/(2y^2 + 3y + 9)]_{y^2=-2y+21} = (29y - 1145)/298.$$

The reader will obtain without difficulty the identity

$$2(x^3 + 1)/X_1^2X_2 = (140x - 10)/149X_1^2$$
$$+ (87x - 1145)/149^2X_1 - (58x - 773)/149^2X_2,$$

where

$$X_1 = 3x^2 + 2x - 7, \quad X_2 = 2x^2 + x + 1.$$

9. In conclusion, a word on Bôcher's *Introduction*. The reference on p. 55 of the Report concerns only §§1-10 of Ch. 6 (not Ch. 4!); partial fractions are not mentioned in the book. But it does not matter in the least if a young mathematician's acquaintance with Bôcher is due to a misapprehension. Written half a century ago, it is a brilliant and fascinating book, and it ought to be compulsory reading for every student of mathematics in the country to-day.

E. H. N.

ON POWERS OF MATRICES WITH ELEMENTS IN THE FIELD OF INTEGERS MODULO 2

By S. M. Cobb

1. INTRODUCTION

A certain type of transformation of a set of numbers may be represented as the multiplication of a vector by a square matrix. Repetition of the operation is equivalent to multiplying the original vector by a power of the matrix. Properties of powers of matrices are thus of considerable importance, and many general properties have been established for some time.[1,2]

The growing use of systems in which the elements have only two states, representable by 0 and 1, has revealed a requirement for a detailed study of matrices whose elements are taken from the field of integers modulo 2 (the field J_2 in the notation of Reference 1). This field is characterized by the equations $x^2 = x$ and $x + x = 0$, so that subtraction is the same as addition. The second property also holds for matrices whose elements are taken from the field, but it is not generally true that powers of a matrix are the matrix itself. This paper is concerned with proving certain properties of powers of such matrices. The first theorem is true for matrices with elements from any number field, but no previous statement of it has been found. The others are all restricted to matrices with elements in the particular field J_2, although they may all be simply extended to the more general field J_p, where p is a prime integer. All the theorems have proved most useful in the design of systems of the type mentioned at the beginning of this paragraph. The complement of an element (i.e. $x + 1$) can be brought into the design by incorporating into the vector representing a set of elements an additional unit element and then arranging the transformation matrix so that this additional element remains constant. The properties of the new transformation matrix are then those of an $(n + 1)$ by $(n + 1)$ matrix instead of n by n.

2. A GENERAL THEOREM FOR A CLASS OF SINGULAR MATRICES

In this section is proved a theorem about the powers of a singular matrix of a certain type. It is useful both on its own account and also because it is used later in the proof of Theorem 5.

Theorem 1

If A is a square matrix with n rows and columns, the elements being taken from any number field, and if $A^s = 0$, but $A^r \neq 0$ for any $r < s$, then $s \leqslant n$.

Proof

Suppose that the theorem is not true and that $s = n + k$, where k is any positive integer.

Thus $A^s = A^{n+k} = 0$, but $A^r \neq 0$ for $r < n + k$.

Since $A^s = 0$, $|A| = 0$, so that at least one latent root is zero.

Hence the characteristic equation $|A - \lambda I| = 0$, has the form

$$\lambda^q \,.\, P_{n-q}(\lambda) = 0,$$

where q is a positive integer and $P_{n-q}(\lambda)$ is a polynomial of order $(n - q)$ in λ.

By the Cayley-Hamilton Theorem[1,3] we have

$$A^q \,.\, P_{n-q}(A) = 0.$$

i.e. A satisfies a polynomial identity of the form

$$a_0A^q + a_1A^{q+1} + a_2A^{q+2} + \ldots + a_{n-q}A^n = 0,$$

where the a_t are not all zero, and where we may assume without loss of generality that $a_0 \neq 0$.

Multiplying through by A^k we have

$$a_0A^{q+k} + a_1A^{q+k+1} + a_2A^{q+k+2} + \ldots + a_{n-q}A^{n+k} = 0.$$

But by the hypothesis $A^s = A^{n+k} = 0$.

$$\therefore \quad a_0A^{q+k} + a_1A^{q+k+1} + a_2A^{q+k+2} + \ldots + a_{n-q-1}A^{n+k-1} = 0.$$

Multiplying through by A we have

$$a_0A^{q+k+1} + a_1A^{q+k+2} + a_2A^{q+k+3} + \ldots + a_{n-q-1}A^{n+k} = 0,$$

which reduces again to

$$a_0A^{q+k+1} + a_1A^{q+k+2} + a_2A^{q+k+3} + \ldots + a_{n-q-2}A^{n+k-1} = 0.$$

Successive reduction eventually yields

$$a_0A^{n+k-1} = 0.$$

But $a_0 \neq 0$, so that $A^{n+k-1} = 0$, contrary to the initial hypothesis.

Hence $s \not> n$.

Clearly $s \leqslant n$ if all the latent roots are zero, so the theorem is proved.

The Auxiliary Unit Matrix of Turnbull and Aitken[2] is a special case of this theorem in which $s = n$.

3. RELATED MATRICES

In this section will be proved two very useful theorems concerning powers of related matrices the elements of which are all taken from the field J_2.

First we prove a well-known lemma.

Lemma

$(I + A)^{2^s} = I + A^{2^s}$, where s is any positive integer or zero.

Proof

The lemma is clearly true if $s = 0$.

Where $s = 1$ we have

$$\begin{aligned}(I + A)^2 &= I^2 + IA + AI + A^2\\ &= I + A + A + A^2\\ &= I + A^2,\end{aligned}$$

since $A + A = 0$ when the elements are from the field J_2. Hence the lemma is true for $s = 1$.

Repetition of the squaring process proves the lemma for general s.

Theorem 2

If s is a positive integer or zero and $A^{2^s} = 0$ then $(I + A)^{2^s} = I$, and conversely.

Proof

$$\begin{aligned}(I + A)^{2^s} &= I + A^{2^s} \text{ by the lemma}\\ &= I, \text{ since } A^{2^s} = 0.\end{aligned}$$

The converse is proved by reversing the above procedure.

Theorem 3

If s is a positive integer and $A^{2^s-1} = I$ and $|I + A| \neq 0$, then $(I + A)^{2^s-1} = I$.

Proof

Since $\quad A^{2^s-1} = I$

then $\quad A^{2^s} = A.$

$\therefore \quad I + A^{2^s} = I + A.$

But $\quad I + A^{2^s} = (I + A)^{2^s}$ by the lemma

$\therefore \quad (I + A)^{2^s} = I + A.$

But $\quad |I + A| \neq 0$, so we may divide both sides by $I + A$

$\therefore \quad (I + A)^{2^s-1} = I$, which proves the theorem.

4. MAXIMUM INDEPENDENT MATRIX POWERS

If the elements of a matrix A with n rows and columns are taken from the field J_2, then it is almost obvious that the maximum

possible power of A which is not a repetition of a lower power is $2^n - 1$, for if A is multiplied by a vector X to produce a different vector Y then zero must be excluded from the 2^n possible vectors X.

The question arises as to whether it is always possible to generate the $2^n - 1$ non-zero vectors simply by successive multiplication by a matrix A with elements from J_2. That this is so is obvious from the general theory of finite fields (Reference 1). For in the root field F of the polynomial $x^q - x$ over J_2, where $q = 2^n$, every primitive $(q-1)$st root of unity, which has no lower power equal to 1, has a unique expression as a solution of an equation of the form $f(x) = 0$, where $f(x)$ is an irreducible polynomial of degree n over J_2. Such a polynomial is just the characteristic polynomial of a matrix A with elements from J_2, so that by the Cayley-Hamilton Theorem A satisfies $f(A) \equiv 0$. Hence we have proved the following theorem.

Theorem 4

It is always possible to find a matrix A with n rows and columns with elements from the field J_2 which has the property $A^{2^n-1} = I$, but that $A^r \neq I$ for any $r < n$, the characteristic function of such a matrix being a polynomial of degree n irreducible over J_2.

Our next theorem concerns powers of A of the form 2^q.

Theorem 5

If A is a matrix such that $A^{2^q} = 1$ for some integral q, but that $A^r \neq I$ for any $r < 2^q$, then the largest value of 2^q is the next integral power of 2 greater than or equal to n.

Proof

We can find a matrix A such that $(I + A)^s = 0$ but $(I + A)^r \neq 0$ for $r < s$, where, by Theorem 1, $s \leqslant n$.

But if $(I + A)^n = 0$, then $(I + A)^{n+1} = 0$, and so does $(I + A)^{2^q}$, where 2^q is the next integral power of 2 greater than or equal to n.

But if $(I + A)^{2^q} = 0$, we have by Theorem 2 that $I = A^{2^q}$, which proves the theorem.

5. THE CHARACTERISTIC FUNCTION OF A SQUARED MATRIX

The final theorem we shall prove concerns the preservation under squaring of the characteristic function of a matrix A with elements from J_2. It is well-known, but is included in order to complete the survey of properties of these matrices.

Theorem 6

Squaring a matrix A with elements from J_2 leaves the characteristic function unchanged.

Proof

Let the characteristic function of A be $f(A)$.

$\therefore$ By the Cayley-Hamilton Theorem $f(A) \equiv 0$.

$\therefore$ $f^2(A) \equiv 0$.

But by similar reasoning to that in the lemma of Section 2 cross-product terms disappear, so that

$$f^2(A) = f(A^2)$$

$$\therefore \quad f(A^2) \equiv 0$$

$$\therefore \quad f(A) \text{ divides } f(A^2).$$

But as both characteristic functions are polynomials of the same degree their ratio can only be a constant, which must be unity since the coefficients are taken from the field J_2. Hence A and A^2 have the same characteristic function.

From this we notice that if A is a matrix such that $A^{2^n-1} = I$ but $A^r \neq I$ for any $r < 2^n - 1$ there are precisely n powers of A with the same characteristic function. But if $\varphi(s)$ is the number of integers less than s which are prime to s (Euler's Function), there are exactly $\varphi(2^n - 1)$ powers of A which are generators of the group of powers, which together with the zero element form the root field F of the polynomial $x^{2^n} - x$ over J_2. There are thus precisely $\dfrac{\varphi(2^n - 1)}{n}$ different characteristic functions corresponding to matrices A which generate all the non-zero elements of F.

6. ACKNOWLEDGEMENTS

This work was carried out as part of the research programme of the Plessey Co. Ltd., Roke Manor, Romsey, and is published with the permission of the Company.

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2. An Introduction to the Theory of Canonical Matrices. H. W. Turnbull and A. C. Aitken. Blackie and Son.

S. M. C.

GLEANINGS FAR AND NEAR

1918. *He's the man with a million ergs.*—Charlie Grace whined on to the stage of the London Hippodrome last night ... Charlie, the latest rock'n roll import, is 21 and only 5ft. 4in. But what he lacks in inches he makes up in ergs, the yardstick of energy. He uses about a million a show.

Daily Mail, Tuesday, Aug. 6, 1957 (Per Mr. D. A. Quadling).

A DEMONSTRATION BINARY ADDER

By H. Martyn Cundy

Nowadays, when so much computing is done with the use of the binary scale, it is arguable that some knowledge of binary arithmetic should be part of the mathematical equipment of the normal grammar-school pupil. The basic electric circuits for binary addition are comparatively simple, and a satisfactory machine for demonstrating the principles can be constructed with standard multiple switches and a flash-lamp battery and bulbs. In the U.S.A. the progressive firm of Berkeley Enterprises, Inc., of Newtonville Mass. has popularized "Geniac" kits from which various simple computing devices can be manufactured from interchangeable components. I am indebted to the Geniac Manual for some of the basic circuits I have used, though I have modified them considerably.

Fundamental Operations

The rules for addition in binary scale are simple: $0 + 0 = 0$, $0 + 1 = 1 + 0 = 1$, $1 + 1 = 10$. If we call the digits to be added a and b, then it appears that the digit to be "written down" in the sum $a + b$ is 0 if a and b are the same and 1 if a and b are different; in the language of symbolic logic it is the value of $a \not\equiv b$ ("a not-equivalent to b"), being 1 when this statement is true and 0 when it is false. The digit to be "carried" however is 1 only when a and b are both 1, otherwise it is 0; it is in fact $a \,.\, b$, the logical product of a and b, which is the same as their product considered as binary digits.

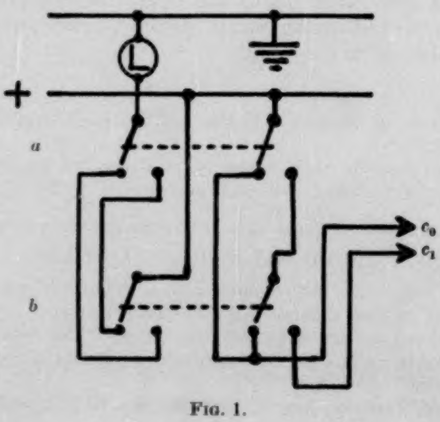


Fig. 1.

Electrical Design

It is easy to design electric circuits in which the values of a and b are represented by the positions of switches and the values of $a \not\equiv b$ and $a \,.\, b$ are shown in lights. The ordinary two-way staircase switch is a "not-equivalent" circuit: the light is "on" when the switches are opposed and "off" when they are both down or both up. Such a circuit is shown in the left half of Fig. 1. (The switches are all shown in the '0' position.) The right half of Fig. 1 is the circuit for the "carry" digit. Current will pass along the line c_1 when both a and b are 1; i.e. c_1 gives the "carry" digit $a \,.\, b$. But in addition a circuit is provided to carry 0, giving current in c_1 when either a or b (or both) is 0; i.e. c_0 gives "not both a and b". The necessity for the c_0 line will appear later.

Fig. 1 thus shows a complete unit for adding two single binary digits, "putting down" the sum in the light L and "carrying" 0 or 1 to c_0 or c_1 respectively. Such a device is called a *half-adder*. Two two-pole two-way switches are required.

Except at the least significant end of two binary numbers which are to be added, there are *three* possible digits which have to be added at each stage, since there may be a "carry in" digit as well as the two digits in the binary numbers. E.g. in the sum

$$\begin{array}{r} 111 \\ 110 \\ \hline 1101 \\ \hline \end{array}$$

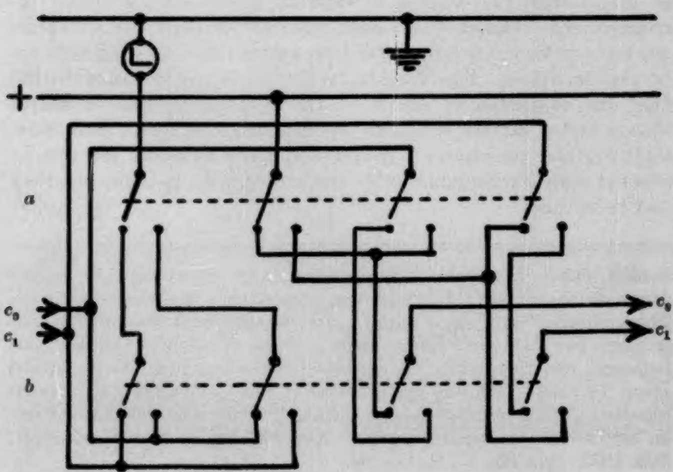


FIG. 2.

the third column from the right (the "fours" column) contains the addition of two 1's and a carry of 1 from the twos column. To effect this without the use of rectifiers four-pole switches are needed, as shown in Fig. 2, though other arrangements are possible. The left hand side of this diagram caters for the "put down" figure indicated by the light; the rest deals with the "carry out" to the next column. Note that these diagrams are shown from the underneath or wiring side of the machine, so that the least significant digit is on the *left*. On the upper (switch) side the digits will be in conventional order.

It is now clear why a c_0 line is required, for we need to bring up power to each stage even if there is 0 to carry. Power injected separately at each stage could get back down the c_1 line to earlier stages unless rectifiers were included. The multiplicity of switches could be reduced if this were done, but an extra pole on a switch is cheaper than a germanium diode or other rectifier. Power has to be injected for the same reason to operate the outgoing c_0 line when the a and b digits are both zero, since otherwise there would have to be a connexion in this condition between the ingoing c_0 and c_1 lines which would allow current to get back into the previous stage.

Hardware Details

To add two numbers from 0 to 15, four binary digits are needed in each number and five in the answer. The least significant digit requires a half adder (Fig. 1), the remainder full adders; except that the most significant full adder needs no c_0 carry-out line and it will be found that two three-pole switches suffice with a minor re-arrangement. Therefore for each number we need one two-pole, one three-pole, and two four-pole two-way switches, costing between 1s. and 2s. apiece. Five flash-bulbs for displaying the answer—the final one connected of course to the outgoing c_1 line—a simple chassis and a battery complete the requirements, apart from wire, soldering and patience. A neater and more compact job can be effected with germanium diodes, dispensing with c_0 lines, but they cost 8s. a time.

H. M. C.

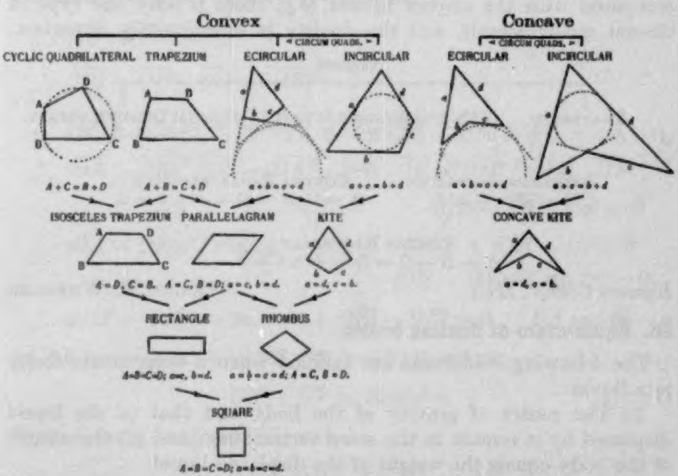
1919. 'The "thousand," incidentally, was rather like a baker's dozen, being actually 1200, and was counted in a most ingenious way. The nailmaker hung up a simple pair of scales and counted 38 nails into one pan. He next poured nails into the other until the two pans balanced, and then added their contents; the one pan now contained about 76 nails. This was again balanced and the doubling-up process repeated until a number approximating to 1200 was reached.'—From an article on the ancient craft of Nail Making in *The Woodworker*, Feb. 1958. [per Mr. W. H. Cozens.]

25. Quadrilaterals

Most teachers are probably in the habit of compiling for their students a family tree for the plane quadrilaterals of the following type:

QUADRILATERAL
↓
TRAPEZIUM
↓
PARALLELOGRAM
↙ ↘
RECTANGLE RHOMBUS
↘ ↙
SQUARE

Possibly the kite also is included, dangling forlornly from its own special branch, and making the tree lop-sided.



When the subject comes up for revision at a later stage, there is an interesting extension of this genealogy which possesses more symmetry and which incorporates satisfactorily not only the kite, but also the isosceles trapezium, the cyclic quadrilateral and the neglected circum quadrilaterals. It exhibits clearly a duality in the choice of nomenclature between angles and sides, though this important feature can only be emphasized by listing properties which are not, in general, the ones we choose to *define* the quadilateral in question.

Although the ordinary circum quadrilateral (which we have for purposes of distinction called incircular) is well-known, the ecircular type is less familiar, and it is rather pleasing to discover that this figure finds a natural place in our convex pedigree opposite the trapezium. Both types of circum quadrilateral, of course, have the noteworthy, but more sophisticated, properties which are associated with the foci of certain conics: an ellipse for the ecircular quadrilateral and 2 hyperbolas for the incircular.

It should be noted for completeness that there is also a similar table for the crossed quadrilaterals, though, except for the contra parallelogram which has some importance as a linkage (Hart's inversor), these are usually considered of less interest. An outline only is given here and the diagrams are left to the reader. If the sides which cross are denoted in the one notation by AB and CD, and in the other by a and c, then $A + D = B + C$ and $a + c > b + d$. These relations limit the number of possibilities compared with the convex figures, (e.g. there is only one type of circum quadrilateral), and the duality is consequently imperfect:

Crossed

TRAPEZIUM $A - B = C - D = 0$ ↘ | CYCLIC QUADRILATERAL $A - C = B - D = 0$ ↙ ↘ | CIRCUM QUADRILATERAL $a - c = b - d$ ↙

ISOSCELES TRAPEZIUM $A = B = C = D$ ↘ | CONTRA PARALLELOGRAM $A = C, B = D; a = c, b = d.$ ↙

CONTRA RECTANGLE $A = B = C = D; a = c, b = d.$

Hymers College, Hull. ROGER F. WHEELER

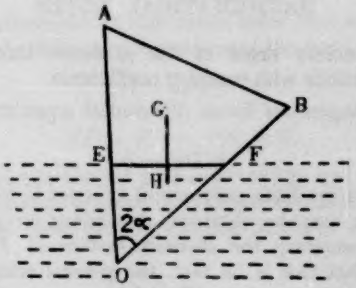
26. Equilibrium of floating bodies

The following conditions are fulfilled when a body floats freely in a liquid:

(i) The centre of gravity of the body and that of the liquid displaced by it remain in the same vertical line, and (ii) the weight of the body equals the weight of the displaced liquid.

The application of the method of vectors, particularly to condition (i), facilitates the solution of certain classical problems as may be judged from the following illustration:

"An isosceles triangular lamina floats with its plane vertical, its vertical angle ($=2\alpha$) immersed and the base wholly above water. Prove that in the position of equilibrium in which the base is not horizontal, the sum of the lengths of the immersed portions of the two sides is $2a \cos^2 \alpha$ and their product $a^2\sigma$, where a is the length of either of the equal sides and σ the density of the lamina."



In the position of equilibrium, let EF be the line of intersection of the plane surface of water and the plane of the lamina OAB. Let G be the centroid of the lamina and H that of the immersed portion. Then, in view of condition (i), GH is vertical and hence perpendicular to EF which is horizontal.

$$\therefore\ \overline{HG}\,.\,\overline{EF}=0$$

$$\therefore\ (\overline{OG}-\overline{OH})\,.\,(\overline{OF}-\overline{OE})=0$$

$$\therefore\ \{\tfrac{1}{3}(\overline{OA}+\overline{OB})-\tfrac{1}{3}(\overline{OE}+\overline{OF})\}\,.\,(\overline{OF}-\overline{OE})=0$$

$$\therefore\ (\overline{OA}-\overline{OE})\,.\,\overline{OF}-(\overline{OA}-\overline{OE})\,.\,\overline{OE}+(\overline{OB}-\overline{OF})\,.\,\overline{OF}$$
$$-(\overline{OB}-\overline{OF})\,.\,\overline{OE}=0$$

$$\therefore\ (OA-OE)OF\cos 2\alpha-(OA-OE)OE+(OB-OF)OF$$
$$-(OB-OF)OE\cos 2\alpha=0$$

$$\therefore\ a(OF-OE)(\cos 2\alpha+1)+OE^2-OF^2=0,\ \because OA=OB=a$$

As AB is not parallel to EF, $OE\neq OF$, therefore

$$OE+OF=2a\cos^2\alpha \qquad \ldots.(1)$$

Further, from condition (ii), we have

$$\tfrac{1}{2}g\sigma\,\overline{OA}\times\overline{OB}=\tfrac{1}{2}g\,.\,\overline{OE}\times\overline{OF}$$

or, $$\sigma\,.\,OA\,.\,OB\sin 2\alpha=OE\,.\,OF\sin 2\alpha$$

$\therefore$ $$OE\,.\,OF=a^2\sigma \qquad \ldots.(2)$$

Lucknow Christian College, India R. S. L. SRIVASTAVA

1920. It is the spin of the Earth—at over a thousand miles an hour between the tropics—that gives a twist to the cyclone.... E. G. R. Taylor, *The Haven-finding Art*, p. 20. [Per Mr. J. C. W. De la Bere.]

2784. An elementary proof of the existence theorem for linear differential equations with constant coefficients

Given a homogeneous linear differential equation with constant coefficients

$$P(D)y = 0,$$

where $P(D)$ is a symbolic polynomial of order n, it is easy to show that a solution may be obtained, depending on n arbitrary constants, by considering the distinct factors of $P(D)$ separately. But to show that this is, in fact, the general solution, it is usual to appeal to the theory of the Wronskian.

An entirely elementary demonstration may, however, be given as follows. Suppose $P(D)$ be decomposed into two polynomial factors, $Q(D)$ and $R(D)$, which have no common factor. Then the given equation is

$$P(D)y \equiv Q(D)R(D)y \equiv R(D)Q(D)y = 0. \tag{1}$$

Since Q and R have no common factor, there exist two other polynomials, $S(D)$ and $T(D)$, such that

$$Q(D)S(D) + R(D)T(D) \equiv 1, \tag{2}$$

where the degree of S is less than that of R, and the degree of T less than that of Q.

Let y_1 be any integral of (1), so that y_1 certainly admits as many derivatives as the degree of P or of QR. Then, noticing that the degrees of QS and of RT are less than that of QR, we may, using (2), write

$$\begin{aligned} y_1 &\equiv \{Q(D)S(D) + R(D)T(D)\}y_1 \\ &\equiv Q(D)S(D)y_1 + R(D)T(D)y_1. \end{aligned} \tag{3}$$

Writing

$$\begin{aligned} v &= Q(D)S(D)y_1, \\ u &= R(D)T(D)y_1, \end{aligned}$$

we see that, since y_1 satisfies (1),

$$\left.\begin{aligned} Q(D)u &\equiv T(D)\{Q(D)R(D)\,.\,y_1\} \equiv T(D)\{P(D)y_1\} = 0, \\ R(D)v &\equiv S(D)\{Q(D)R(D)\,.\,y_1\} \equiv S(D)\{P(D)y_1\} = 0. \end{aligned}\right\} \tag{4}$$

It follows, therefore, from (3) and (4), that every solution of (1) is the sum of a solution of

$$Q(D)y = 0$$

and a solution of

$$R(D)y = 0.$$

Repeated applications of this result show that every solution of (1) is the sum of solutions of equations of the form

$$(D-\alpha)^r y = 0$$

and

$$\{(D-\beta)^2+\gamma^2\}^s y = 0,$$

corresponding to the distinct factors of $P(D)$: and the form of the most general solutions of these last equations is established directly. The form of the most general solution of (1) is thus established.

D. H. PARSONS

2785. Particular integrals of linear partial differential equations with constant coefficients

1. Consider first an equation of the form

$$P(D_1, \ldots, D_m)z = p_n(x_1, \ldots, x_m) \tag{1}$$

($D_i z \equiv \partial z/\partial x_i$, $i = 1, \ldots, m$), where the symbolic polynomial $P(D_1, \ldots, D_m)$ contains a non-zero constant term, and where $p_n(x_1, \ldots, x_m)$ is a polynomial of degree n in $x_1, \ldots, x_m$. To obtain a particular integral, the well-known extension of the usual D-operator method can be used: and this may be justified as follows.

Since P contains a non-zero constant term, we can, by long division, expand $1/P$ in a multiple power series in $D_1, \ldots, D_m$, until the remainder contains no term of degree less than $n+1$. Thus it is easy to see (by induction), that we can construct, by division or otherwise, an identity

$$1 \equiv P(D_1, \ldots, D_m)Q_n(D_1, \ldots, D_m) + R_{n+1}(D_1, \ldots, D_m), \tag{2}$$

where Q_n and R_{n+1} are polynomials in $D_1, \ldots, D_m$, and where R_{n+1} contains no term of degree less than $n+1$. The equation (1) may then be written

$$P(D_1, \ldots, D_m)z = P(D_1, \ldots, D_m)\{Q_n(D_1, \ldots, D_m)p_n\},$$

since

$$R_{n+1}(D_1, \ldots, D_m)p_n \equiv 0.$$

A particular integral of this equation is clearly

$$z = Q_n(D_1, \ldots, D_m)p_n(x_1, \ldots, x_m):$$

and we may write this integral as

$$z = \frac{1}{P(D_1, \ldots, D_m)}\,p_n(x_1, \ldots, x_m),$$

it being understood that we expand $1/P$ in a multiple power series, until the remainder annuls p_n.

2. Suppose next that $P(D_1, \ldots, D_m)$ does not contain a non-zero constant term: or, alternatively, suppose that the function on the right is a polynomial in one or several of the variables, the coefficients being continuous functions of the other variables. For simplicity of writing, let us suppose that there are two independent variables, x and y, and let $\partial z/\partial x \equiv Dz$, $\partial z/\partial y \equiv Ez$. Consider first the equation

$$P(D, E)z \equiv \{P_0(E) + DP_1(E) + \ldots + D^s P_s(E)\}z = a_r(y)x^r, \quad (3)$$

where $a_r(y)$ is continuous. We may suppose $P_0(E) \not\equiv 0$: for if $P_0(E) = P_1(E) = \ldots = P_{i-1}(E) \equiv 0$, but $P_i(E) \not\equiv 0$, we simply treat $D^i z$ as the dependent variable.

The well-known formal extension of the D method is to expand $1/P(D, E)$ in ascending powers of D as far as the term in D^r, the coefficients being rational fractions in E, and then to carry out the indicated differentiations with respect to x, and the indicated inverse operations on $a_r(y)$. But the justification of the method of § 1 no longer applies, since we have no polynomial identity analogous to (2): and the following example shows how wrong results may be obtained. Consider the equation

$$(1 + E^2 - D)z = -8x \sin y. \quad (4)$$

Proceeding formally, we write

$$\begin{aligned} z &= \frac{1}{1 + E^2 - D}(-8x \sin y) \\ &= \frac{-8}{1 + E^2} \cdot \frac{1}{1 - D/(1 + E^2)} x \sin y \\ &= \frac{-8}{1 + E^2}\left\{1 + \frac{D}{1 + E^2}\right\} x \sin y \\ &= 4x \cdot \frac{1}{1 + E^2}(-2 \sin y) + \frac{1}{(1 + E^2)^2}(-8 \sin y) \\ &= 4xy \cos y + y^2 \sin y: \end{aligned}$$

and it is easily verified that this is a wrong result, not being an integral of (4).

To put the method right, we notice that, though $1/P$ cannot be expanded in powers of D, with coefficients which are polynomials in E, we can expand $\{P_0(E)\}^{r+1}/P(D, E)$ in this way, as far as the term in D^r, the remainder being divisible by D^{r+1}. Thus it is easily

proved by induction that, by long division or otherwise, we can construct an identity

$$\{P_0(E)\}^{r+1} \equiv \{P_0(E) + \ldots + D^s P_s(E)\} \cdot$$
$$\cdot \{Q(E) + \ldots + D^r Q_r(E)\} + D^{r+1} R(D, E). \tag{5}$$

where each $Q_0, \ldots, Q_r$ is a polynomial in E, and R a polynomial in D and E. Suppose then that we find a particular integral of the equation

$$\{P_0(E)\}^{r+1} A(y) = a_r(y). \tag{6}$$

Then, (3) may be written as

$$P(D, E)z = \{P_0(E)\}^{r+1}\{A(y)x^r\},$$

or, using (5), and noticing that $D^{r+1}\{A(y)x^r\} \equiv 0$,

$$P(D, E)z = P(D, E)[\{Q_0(E) + DQ_1(E) + \ldots + D^r Q_r(E)\}\{A(y)x^r\}],$$

a particular integral of which is clearly

$$z = \{Q_0(E) + DQ_1(E) + \ldots + D^r Q_r(E)\} . \{A(y)x^r\}. \tag{7}$$

Writing the particular integral of (6) as

$$A(y) = \frac{1}{\{P_0(E)\}^{r+1}} a_r(y),$$

we see that the integral (7) of the given equation (3), may be written in the form

$$z = \frac{\{P_0(E)\}^{r+1}}{P(D, E)} \left[\frac{1}{\{P_0(E)\}^{r+1}} a_r(y)x^r\right] \tag{8}$$

the convention being made that the operation

$$\frac{1}{\{P_0(E)\}^{r+1}}$$

is carried out first, and the fraction $\{P_0(E)\}^{r+1}/P(D, E)$ then expanded in ascending powers of D as far as the term in D^r. The equation

$$P(D, E)z = \sum_{r=0}^{n} a_r(y)x^r$$

is now dealt with by treating the terms on the right separately, by the foregoing method.

Example 1

$$(E - D - D^2)z = 132xy^{10} + 504x^2y^6.$$

We first consider the equations

$$E^2 f(y) = 132y^{10}$$

$$E^3 g(y) = 504y^6,$$

which are satisfied by $f = y^{12}$, $g = y^9$. The given equation then becomes

$$(E - D - D^2)z = E^2(xy^{12}) + E^3(x^2y^9).$$

$$\therefore \qquad z = \frac{E^2}{E - D - D^2}(xy^{12}) + \frac{E^3}{E - D - D^2}(x^2y^9)$$

$$= (E + D) \, . \, xy^{12} + \{E^2 + ED + (E + 1)D^2\}(x^2y^9)$$

$$= 18y^8 + 2y^9 + y^{12} + 6x(3y^8 + 2y^{11}) + 72x^2y^7.$$

Example 2

$$(1 + E^2 - D)z = -8x \sin y$$

(the equation considered earlier).

We first consider

$$(1 + E^2)^2 f(y) = -8 \sin y;$$

and, by the usual method, we find the integral

$$f(y) = y^2 \sin y.$$

The given equation then becomes

$$(1 + E^2 - D)z = (1 + E^2)^2(xy^2 \sin y)$$

$$\therefore \qquad z = \frac{(1 + E^2)^2}{1 + E^2 - D}(xy^2 \sin y)$$

$$= (1 + E^2 + D)(xy^2 \sin y)$$

$$= y^2 \sin y + 2x(\sin y + 2y \cos y).$$

3. Reverting to the equation (3), it sometimes happens that we can construct, instead of the identity (5), an identity of the form

$$\{P_0(E)\}^p = \{P_0(E) + DP_1(E) + \ldots + D^s P_s(E)\} \, .$$
$$. \, \{S_0(E) + \ldots + D^r S_r(E)\} + D^{r+1} T(D, E)$$

where each $S_0, \ldots, S_r$ is a polynomial in E, T is a polynomial in D and E, and $p < r + 1$. If this be so, by reasoning entirely similar to the foregoing, we have an integral

$$z = \frac{\{P_0(E)\}^p}{P(D, E)} \left[\frac{1}{\{P_0(E)\}^p} a_r(y) x^r, \right]$$

with the convention that the operation $\dfrac{1}{\{P_0(E)\}^p}$ is carried out first,

and then $\{P_0(E)\}^p/P$ is expanded in ascending powers of D. This integral is thus simpler than the integral (8), and involves simpler calculations.

Example 3

$$(E - D^2)z = 132xy^{10} + 504x^2y^6.$$

Noticing that $(E - D^2)$ does not contain D^1, we see that the expansions of $E/(E - D^2)$ and $E^2/(E - D^2)$ can be carried as far as the terms in D and D^2 respectively, without introducing non-polynomial coefficients (in passing, we mention that this corresponds to the identities

$$E \equiv (E - D^2) + D^2$$

and

$$E^2 \equiv (E - D^2)(E + D^2) + D^4).$$

Thus considering first the equations

$$Ef(y) = 132y^{10},$$
$$E^2g(y) = 504y^6,$$

we have the integrals $f(y) = 12y^{11}$, $g(y) = 9y^8$. The given equation thus becomes

$$(E - D^2)z = E(12xy^{11}) + E^2(9x^2y^8),$$

and we write

$$\begin{aligned} z &= \frac{E}{E - D^2}\,12xy^{11} + \frac{E^2}{E - D^2}\,(9x^2y^8) \\ &= (1)\,.\,12xy^{11} + (E + D^2)9x^2y^8 \\ &= 12xy^{11} + 72x^2y^7 + 18y^8. \end{aligned}$$

D. H. Parsons

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Wokingham, Berks.

2786. On the coincidence of the instantaneous centre and the centre of no acceleration

In note 2711 it is shown that when a lamina moving in its own plane is instantaneously at rest, the point of the lamina which then coincides with the instantaneous centre has zero acceleration. The purpose of this note is to give a shorter proof of the same result.

With respect to axes fixed in the plane of motion let the coordinates of a point fixed in the lamina and the instantaneous centre be (x, y) and (a, b) respectively. Let the angular velocity of the lamina be ω. Then, if $z = x + iy$ and $c = a + ib$,

$$\dot{z} = i(z - c)\omega$$

(since the instantaneous motion of the lamina is one of rotation

about the instantaneous centre with angular velocity ω). Therefore

$$\ddot{z} = i(\dot{z} - c)\omega + i(z - c)\dot{\omega}, \tag{1}$$

and so $\ddot{z} = 0$ when $\omega = 0$ and $z = c$.

This proves the result, but two points require further discussion:

(i) the definition of the position of the instantaneous centre when the lamina is instantaneously at rest;

(ii) the conditions for the validity of the result.

Let the lamina be instantaneously at rest at time t_0, and suppose $\omega \neq 0$ for all t sufficiently near to but not equal to t_0. Then as regards (i) it is assumed that

$$c \to \text{a limit } c_0 \text{ as } t \to t_0,$$

and the position of the instantaneous centre at time t_0 is defined to be c_0.

With this definition the result then holds if $\dot{c}$, $\dot{\omega}$ exist at t_0.

More generally, it holds if c, ω have derivatives on the right and on the left at t_0; for then (1) (regarded as an equation connecting the one-sided derivatives of $\dot{z}$, z, c and ω) shows that, if $z = c$, $\dot{z}$ has right- and left-derivatives both equal to zero at t_0, and so $\ddot{z}$ exists and equals zero at t_0. However, I find it difficult to construct an actual motion in which c has different non-zero right- and left-derivatives.

L. E. Clarke

University College of Ghana

2787. Coloured cubes: a correction to Note 2736

The second-last paragraph of Note 2736 requires correction. For the set of cubes described, the network diagram does *not* allow only one choice of mutually exclusive sets of lines, as stated. The true situation here is in some ways more interesting still.

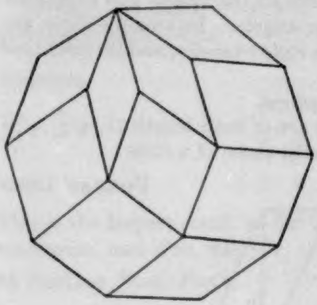
The complete set of network lines can in fact be distributed (in only one way) into *three* sub-sets, such that *any two* of these form the basis of a solution. As a result there are three basic solutions, and the probability of success on random assembly is 5/10,368—five times higher than I stated before.

If we repeat the solution diagram apart from interchanging A and B on the horizontal axis for the extreme right-hand cube (equivalent to rotating this cube about the axis joining the C and D faces), this one diagram then serves to exhibit all three solutions, if we assume the cubes taken as shown and aligned by translation only, without rotation, along the three possible axes in succession.

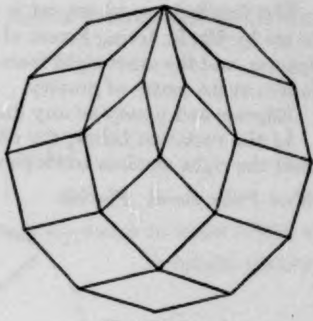
Detection of my oversight derives from an empirical discovery of one of the additional solutions by Mr. L. G. Will of Dornoch, Sutherland.

Thos. H. O'Beirne

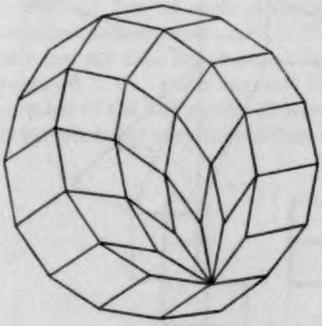
2788. Polygon dissections.



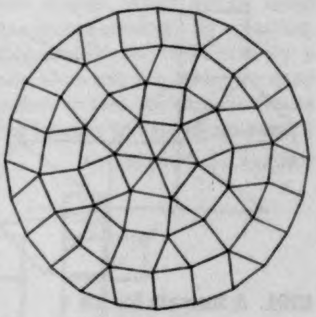
The 10-gon



The 12-gon



The 16-gon



The 24-gon

C. DUDLEY LANGFORD

2789. A formula for π

$$\pi = 32 \tan^{-1}\left(\frac{1}{10}\right) - 16 \tan^{-1}\left(\frac{1}{515}\right) - 4 \tan^{-1} \frac{1}{239}.$$

Fifteen terms of the series for the first inverse tangent, with six terms each of the series for the second and third give π to thirty places of decimals.

G. F. FREEMAN

Woolwich Polytechnic

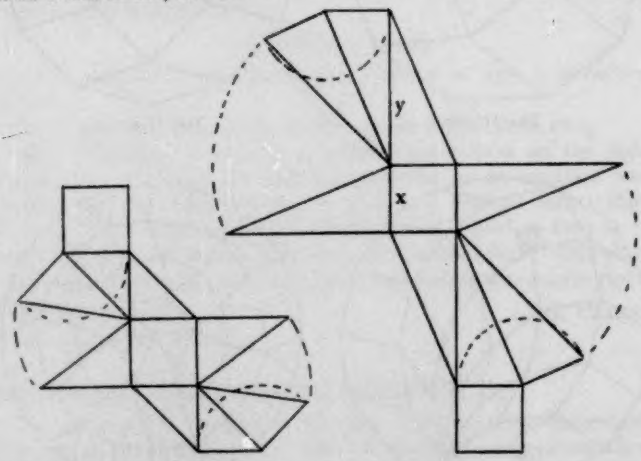
2790. A unique decahedron

The decahedron, whose net is shown in the figure, was suggested to me by Mr. E. Irving Freese of Los Angeles. Its two end faces are *squares*, and the other eight faces are *right triangles*, and its thickness is *zero* at its centre of gravity.

Edges x and y may be any dimensions.

In the variation below, the edges are of such length $(1, \sqrt{2}, \sqrt{3})$ that the eight vertices correspond with those of a cube.

West Palm Beach, Florida DORMAN LUKE



2791. A formula for log t

It will be found that the formula

$$\log_e t = 6(t-1)/(t+1+4\sqrt{t})$$

reproduces the Napierian logarithm of t with almost five figure accuracy over the extensive range $1 < t < 2$.

The formula is easily established by applying Simpson's rule of approximate integration to the integral $\int_0^1 t^x\,dx$.

The high accuracy is probably due to the fact that the remainder in Simpson's rule, expressed as an expansion in ascending powers of the difference operator over a unit interval, begins with a multiple of $\Delta^4 - \frac{3}{2}\Delta^5$, which vanishes for the function $f(x) = \left(\frac{5}{3}\right)^x$; noting $1 < \frac{5}{3} < 2$.

S. J. TUPPER

2792. Problem in determinants

Arrange the digits from 1–9, to form a determinant giving the greatest possible value.

Example

$$\begin{vmatrix} 1 & 4 & 8 \\ 7 & 2 & 6 \\ 5 & 9 & 3 \end{vmatrix} = 412$$

This is the largest result so far but nobody seems to know if it is a maximum, and if so, why?

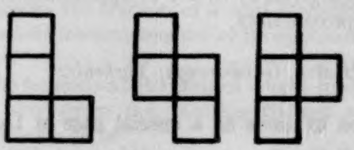
SINCLAIR GRANT

83 *Needless Road, Perth.*

2793. A conundrum for form VI

Cut out four of each of these three shapes. They can be fitted together to make a square 7 × 7 with a centre missing. By counting, one can see that any 9 taken together contain the right area for a square 6 × 6 ... yet it can only be done when one has taken an even number of the last piece! (I leave readers to find the explanation as it is reasonably easy to find, but rather lengthy to explain on paper.)

C. DUDLEY LANGFORD



2794. A bogus experiment

The experiment of dropping a penny with a small piece of paper on it is strongly recommended in the Association's Report on the teaching of Mechanics and has been included in several text-books. Those who accept this experiment as showing that the gravitational acceleration is the same for all bodies may like to modify it as follows.

On top of the paper (which can be nearly as large as a halfpenny) is placed a small dense object such as a meccano bolt; the penny is held horizontal and then moved downwards *by hand* with an acceleration greater than g. It will be found that the bolt is left behind, but the paper still goes with the penny.

A. BARTON

Cheltenham College

2795. Merial triangles

Let a triangle be said to be *merial* if the square on one side is equal in area to the rectangle contained by the other two. It follows from the definition that a triangle similar to a merial triangle is itself merial, and a triangle whose sides are the products of corresponding sides of merial triangles is also a merial triangle.

Consider an acute angled triangle ABC, with $a < b < c$. Let D be the reflection of C in the perpendicular from B to AC and let the circle BCD cut AB again in E. Then, if

(i) b/a is the real root of the equation $x^5 = x + 1$,

(ii) $c/a = (b/a)^2$,

all the six triangles ABC, ADE, BCE, ABD, ACE, and BDE are merial triangles.

The triangles ABC, ADE and CBE are similar and therefore all merial since ABC is merial by condition (ii). Since $AD \,.\, AC = BA^2 - BD^2$ therefore $AD/a = (c^2 - a^2)/ab$; let $b/a = x$, where x is the real root of $x^5 = x + 1$, then $c/a = x^2$ and

$$AD/a = \{(c/a)^2 - 1\}(a/b) = 1/x^2,$$

and so the sides of the triangle ABD are proportional to 1, $1/x^2$, x^2 proving that ABD is a merial triangle. Since ACE and BDE are similar to ABD, all these are merial triangles.

The value of b/a given by (i) is 1·17 approximately and the value of c/a is 1·36 approximately.

E. PEASE

Lowcross Gate, Hulton, Guisborough, Yorkshire

2796. Integration by parts as a special case of Leibnitz' Theorem

If we write D for the operator d/dx, and D^{-1} for the inverse operator $\int \ldots dx$ the rule for integration by parts can be expressed as

$$D^{-1}(uv) = u \,.\, D^{-1}v - D^{-1}\{Du \,.\, D^{-1}v\}.$$

If u is a polynomial in x of degree K, repeated application gives

$$D^{-1}(uv) = u \,.\, D^{-1}v - Du \,.\, D^{-2}v + D^2u \,.\, D^{-3}v - \ldots$$

where the series on the right terminates at the $(k + 1)th$ term. This is Leibnitz' Theorem with $n = -1$. For example

$$\int (x^3 + 2x)e^{2x}\,dx = (x^3 + 2x) \,.\, \frac{e^{2x}}{2} - (3x^2 + 2) \,.\, \frac{e^{2x}}{4} + 6x \,.\, \frac{e^{2x}}{8} - 6 \,.\, \frac{e^{2x}}{16}\,.$$

The College of Technology, Huddersfield C. V. GREGG

2797. A construction problem

Mr. G. Ash has raised the interesting question of whether it is possible to construct (by Euclidean methods) a triangle when the distances of the vertices from the ortho-centre are given.

Let these distances be u, v, w. Then, in the usual notation,

$$u = 2R\cos A,\ v = 2R\cos B,\ w = 2R\cos C.$$

Using the identity

$$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C = 1,$$

we obtain

$$4R^3 - R(u^2 + v^2 + w^2) - uvw = 0. \tag{1}$$

The classical problem of trisecting an angle leads to the equation

$$4x^3 - 3x - c = 0 \tag{2}$$

where c is a given constant. Since (1) can be reduced to (2) by a simple transformation, and since the trisection of an angle is known to be, in general, impossible (by Euclidean methods), it follows that the given construction is, in general, impossible.

E. J. F. PRIMROSE

2798. A note on belt friction

This note concerns the friction of a cord stretched over a drum. This is known as belt friction because of its application to belts and pulleys.

The ratio of the tensions at the ends of a light cord just producing sliding of cord over drum (i.e. limiting friction at all points of contact) is well known to be

$$\frac{T_1}{T_2} = e^{\mu\theta}$$

where

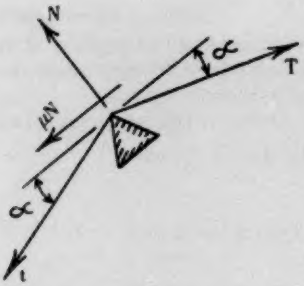
θ = angle of wrap

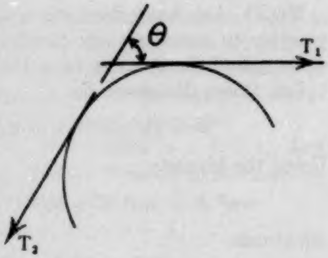
and

μ = coefficient of friction between cord and drum.

This expression is usually derived by considering the forces on an element of the cord and integrating over the angle of wrap. A different method of derivation may be demonstrated as follows:

Consider first a knife edge with the light cord stretched symmetrically over it. Let the tensions in the cord on either side of the knife edge be T and t ($T > t$), and let these be so adjusted as to cause the cord just to slip over the knife edge. Let N be the normal reaction of knife edge on cord and let μ be the relevant coefficient





of friction. Then the forces on the cord are as shown in the figure and

$$(T + t) \sin \alpha = N$$

$$(T - t) \cos \alpha = \mu N,$$

hence

$$\frac{T}{t} = \left[\frac{\cos \alpha + \mu \sin \alpha}{\cos \alpha - \mu \sin \alpha}\right]$$

Next, consider the cord to be stretched over n such knife edges. Let the tensions in the cord on either side of the system of knife edges be T_1 and T_2, $(T_1 > T_2)$, so that the cord just slips over the knife edges. Then

$$\frac{T_1}{T_2} = \left[\frac{\cos \alpha + \mu \sin \alpha}{\cos \alpha - \mu \sin \alpha}\right]^n$$

If, in wrapping the cord over the knife edges one end of the cord has been turned through an angle θ relative to the other end, then

$$\theta = 2n\alpha \quad \text{i.e.} \quad \alpha = \frac{\theta}{2n}.$$

This system of knife edges will approach a continuous surface as $n \to \infty$, and since

$$\frac{\log (1 + x)}{x} \to 1 \text{ as } x \to 0,$$

therefore

$$\log \{1 + \mu \tan \alpha\}^n = \frac{\log \{1 + \mu \tan \alpha\}}{\mu \tan \alpha} \cdot \frac{\mu \tan \alpha}{\alpha} \cdot \frac{\theta}{2} \to \tfrac{1}{2}\mu\theta$$

and

$$\log \{1 - \mu \tan \alpha\}^n \to -\tfrac{1}{2}\mu\theta,$$

so that

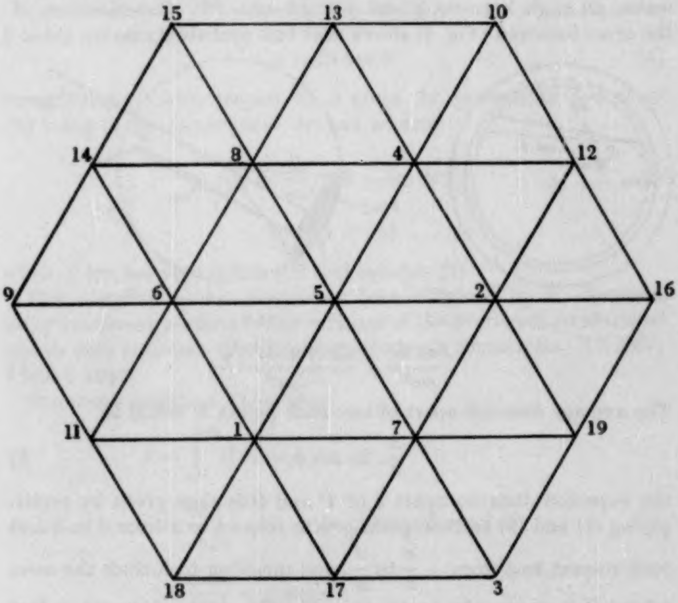
$$\lim_{n \to \infty} \frac{T_1}{T_2} = e^{\mu\theta}$$

Neither method of derivation assumes the drum over which the cord is stretched to be circular in section although most of the applications of the theory involve circular profiles.

O. R. Fendrich

Battersea College of Technology

2799. Magic hexagon



The sum of the numbers on any line equals 38.

T. Vickers

1, Hospital Bridge Rd., Twickenham

1921. *Classical, Intuitionistic, and now ... !* "This type of formal logic 'which is independent of all dramatic or narrative element' is, of course, in the most complete contrast to the formal logic of the Romantic school." Constant Lambert, in "*Music Ho!*" p. 92 "Pelican" edition [per Mr. J. Hooley].

2800. On Note 2754—a repeated integral

In Note 2754 H. G. ApSimon obtains the average distance between two independent random points P, Q in a circle radius a by means of a quadruple integral. This can be obtained by either of two rather simpler methods as follows.

The first is to obtain the joint probability that the first point, say P, is at a distance of between x and $x + dx$ from the centre C, while Q is nearer the centre than P in such a position that PQ makes an angle between ϕ and $\phi + d\phi$ with PC. Consideration of the areas involved (Fig. 1) shows that this probability is

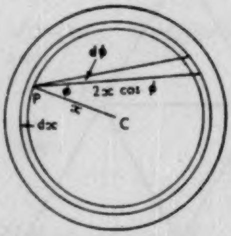


Fig. 1.

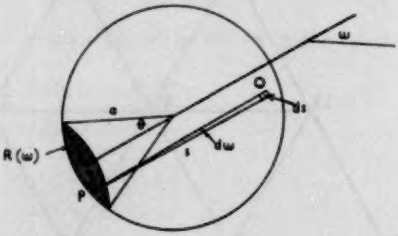


Fig. 2.

$$\frac{2\pi x\, dx}{\pi a^2} \times \frac{(2x \cos \phi)^2\, d\phi}{2\pi a^2} \tag{1}$$

The average distance apart of two such points P and Q is

$$\frac{2}{3} \,.\, 2x \cos \phi; \tag{2}$$

the expected distance apart s of P and Q is then given by multiplying (1) and (2) and integrating with respect to x from 0 to a and with respect to ϕ from $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$, and doubling to include the cases where P is nearer the centre than Q. The double integral in fact involves the product of simple integrals of x^4 and of $\cos^3 \phi$, and the answer is

$$s = \frac{128a}{45\pi}$$

as obtained by ApSimon.

The other method is to obtain the expectation of s from its frequency distribution. If P is the point first chosen, then the line PQ makes an angle with a fixed direction which is uniformly distributed over the range $0, 2\pi$. If the angle is in the range ω, $\omega + d\omega$ and the length PQ is in the range s, $s + ds$ then P must be in the lens-shaped area $R(\omega)$ (see Fig. 2) between the circle and

one displaced relatively to it by the vector QP, and Q must occupy the element of area $s\,ds\,d\omega$.

The joint probability of these events is

$$\frac{Rs\,ds\,d\omega}{\pi^2 a^4} \tag{3}$$

where

$$R = \text{area of } R(\omega)$$

$$= a^2\,(2\theta - \sin 2\theta),$$

θ being defined by

$$s = 2a \cos \theta \tag{4}$$

Integrating (3) with respect to ω gives the probability $f(s)\,ds$ of PQ being in the range s, $s + ds$, and we find

$$f(s) = \frac{2s}{\pi a^2}\,(2\theta - \sin 2\theta) \tag{5}$$

$$(0 < s < 2a)$$

where θ lies between 0 and $\pi/2$ and satisfies (4).

This distribution has previously been obtained by F. Garwood using a different method ("The variance of the overlap of geometrical figures with reference to a bombing problem", Biometrika, XXXIV, 1 and 2 1947).

The mean length of PQ is thus

$$\bar{s} = \int_0^{2a} s f(s)\,ds$$

$$= \frac{8a}{\pi}\int_0^{\pi/2} (2\theta - \sin 2\theta) \sin 2\theta \cos \theta \,d\theta$$

$$= \frac{128a}{45\pi} \doteqdot 0{\cdot}905a$$

It is of interest to note that the mode and the median of the distribution (5) are:—

$$\text{mode} \doteqdot 0{\cdot}835a$$

$$\text{median} \doteqdot 0{\cdot}894a$$

It can readily be verified that the mean of PQ^2 is a^2, from which it follows that the standard deviation of s is $0{\cdot}425a$ approximately.

F. GARWOOD and J. C. TANNER

Road Research Laboratory
Harmondsworth, Middlesex

2801. On Note 2530

I should like to make a remark in connection with Mr. A. P. Rollett's Note 2530 (Math. Gazette 39, 1955, p. 209).

The minimum value of the perimeter of the hexagon (and that of the pentagon also) occurs not in the case $\measuredangle QPP' = \frac{\pi}{3}$ and consequently $\measuredangle OQA = \frac{5\pi}{12}$, but in the case $\measuredangle OQA = \frac{\pi}{3}$ and consequently $\measuredangle QPP' = \frac{5\pi}{12}$. Indeed, the value of the perimeter in the mentioned case is

$$2a(2\sqrt{6} + \sqrt{3} - 2\sqrt{2} - 1) = a \,.\, 5{\cdot}605\ldots,$$

while according to the present correction this value is

$$2a(\sqrt{3} + 1) = a \,.\, 5{\cdot}464\ldots.$$

The area of the pentagon is $a^2/2$.

With reference to Mr. C. D. Langford's letter connected with the Note mentioned above (Math. Gazette 40, 1956, p. 97), I think that the shape drawn is advantageous for tiles from the standpoint of the equality of the sides, however, it is not so versatile because of the cumbersome angles 114° and 131°. On the other hand, in the case of the shape with minimum perimeter the angles are well usable: 90° and 120°, but one of the sides is shorter than the four others: $b(\sqrt{3} - 1)$, b. In both cases something does not go on all fours.

Institut Bolyai de l'Université T. Bakos
Szeged, Hungary

2802. The moments of inertia of the nedian triangles and the nedian hexagon of a plane triangle about a line through the common centre of gravity perpendicular to the plane

A sequel to Note 2734. Further calculation yields the following results ($I/A = k^2$).

Figure	Area	k^2 about GZ
Parent Triangle	$\triangle ABC$	$(a^2 + b^2 + c^2)/36$
Nedian Triangles	$\dfrac{(N-2)^2}{N^2 - N + 1}\triangle ABC$	$\dfrac{(N-2)^2}{N^2 - N + 1}$ (k^2 of $\triangle ABC$)
Nedian Hexagon	$\dfrac{2(N-2)^2}{(N+1)(2N-1)}\triangle ABC$	$\dfrac{(N-2)^2(4N^2 - N + 1)}{(N+1)^2(2N-1)^2}$ (k^2 of $\triangle ABC$)

University of Toronto John Satterly

2803. Limits of iterated logarithmic functions

Let $b_0 = 1$, $b_{n+1} = b^{b_n}$ and $L^0x = x$, $L^{n+1}x = \log_b L^nx$. Then we prove first

THEOREM 1. *If* $1 \leq b \leq e^{1/e}$ *then there is a number* β *in the interval* $(1, e)$ *such that* $b_n \longrightarrow \beta$; *if* $b > e^{1/e}$, $b_n \longrightarrow \infty$.

THEOREM 2. *If* $1 \leq b \leq e^{1/e}$ then L^nn *converges and there is a number* $B \geq e$ *such that, for* $x > \beta$ $L^nx \longrightarrow B$.

A part of *Theorem* 1 was given by Mr. E. de St. Q Isaacson in note 2285 and is included for its application in the proof of *Theorem* 2.

For $x > 0$ the function $x^{1/x}$ steadily increases up to its maximum value $e^{1/e} = 1{\cdot}445$ approximately, at $x = e$, and then steadily decreases towards unity. Hence to any b, $1 < b < e^{1/e}$, there corresponds a unique β, $1 < \beta < e$, and a unique $B > e$ such that $\beta^{1/\beta} = B^{1/B} = b$. Since $b^\beta > b$ therefore $\beta > b$. It follows that $1 < b_n < \beta$ for all $n \geq 1$, for if $1 < b_p < \beta$ for some p then $1 < b^{b_p} < b^\beta = \beta$ so that $1 < b_{p+1} < \beta$.

Consequently

$$b_n^{\ 1/b_n} < \beta^{1/\beta} = b$$

so that

$$b_{n+1} = b^{b_n} > b_n$$

which proves that the sequence b_n is strictly increasing and bounded above by β. Accordingly b_n tends to a limit $l \leq \beta$. Since $b_{n+1} = b^{b_n}$ therefore $l = b^l$ and so $l^{1/l} = b$, which shows that $l = \beta$ since β is the *unique* solution of the equation $x^{1/x} = b$ for $x < e$. In particular if $b = e^{1/e}$, $b_n \longrightarrow e$.

If however $b > e^{1/e}$ then $b > x^{1/x}$ for all $x > 0$ so that $b^x > x$; it follows that in this case too b_n is strictly increasing and so tends to a limit, finite or infinite. Since $b^x > x$ there can be no finite l such that $b^l = l$ and therefore $b_n \longrightarrow \infty$, which completes the proof of Theorem 1.

As examples of this theorem we note:

the sequence $\sqrt{2}$, $(\sqrt{2})^{\sqrt{2}}$, $(\sqrt{2})^{(\sqrt{2})^{(\sqrt{2})}}$, ... tends to 2, and the sequence $\sqrt[3]{3}$, $(\sqrt[3]{3})^{\sqrt[3]{3}}$, $(\sqrt[3]{3})^{(\sqrt[3]{3})^{(\sqrt[3]{3})}}$, ... tends to 2·5 approximately.

Since $L^nx = \log_b \log_b \ldots \log_b x$ with n logarithms to the base b, it follows that $L^nb_n = 1$. Let $1 < b \leq e^{1/e}$, (with $\beta = B = e$ if $b = e^{1/e}$) and $x > B$. Then $L^nx > B$ for all n, for if $L^px > B$ for some p, then $L^{p+1}x = \log_b L^px > \log_b B = B$. Furthermore, since $y > B$ implies $y^{1/y} < B^{1/B} = b$, (i.e. $\log_b y < y$) therefore

$$L^{p+1}x = \log_b L^px < L^px$$

so that L^nx is steadily decreasing and tends to a limit $\lambda \geq B$. Since $\log_b L^nx = L^{n+1}x$ therefore $\log_b \lambda = \lambda$, whence $b = \lambda^{1/\lambda}$

which proves that $\lambda = B$, the unique solution of $x^{1/x} = b$ for $x \geq e$. If $\beta < y < B$ then $\beta < \log_b y < B$ and $y^{1/y} > \beta^{1/\beta} = b$ so that $\log_b y > y$; it follows that if $\beta < x < B$ then $L^n x$ is strictly increasing and bounded above by B, and as before we may show that $L^n x \longrightarrow B$. Of course if $x = B$, $L^n x = B$, and if $x = \beta$, $L^n x = \beta$, which completes the proof that for $x > \beta$, $L^n x \longrightarrow B$. For $x < \beta$, $x < b_p$ for some p and therefore $L^p x < L^p b_p = 1$ so that $L^n x$ does not exist for $n > p + 1$.

To complete the proof of Theorem 2 we note that $(n + 1)^{1/n} \longrightarrow 1$ so that, for $b > 1$, $(n + 1)^{1/n} < b$ for all sufficiently great n, and therefore

$$L^{n+1}(n + 1) = L^n \log_b (n + 1) < L^n n$$

which shows that $L^n n$ is strictly decreasing; since $L^n n > B$ for $n \geq 3$, it follows that $L^n n$ converges to a finite limit.

If $b > e^{1/e}$ then by Theorem 1, $b_n \longrightarrow \infty$ and so, for any x, $b_p > x$ for some p and therefore $L^n x$ does not exist for $n > p + 1$.

There remains to be considered the case $0 < b < 1$, which is the subject of the next note.

PETER GOODSTEIN

2804. The system of equations $b^x = x_1, b^{x_1} = x_2, \ldots, b^{x_n} = x$

If $0 < b < 1$ a consideration of the behaviour of the sequence $b_0 = 1$, $b_{n+1} = b^{b_n}$ leads to the equation $b^{b^x} = x$, which is equivalent to the pair of equations $b^x = y$, $b^y = x$. For we have in turn $b_1 < b_2$, $b_1 < b_3 < b_2$, $b_1 < b_3 < b_4 < b_2$; for some p, let

$$b_1 < b_3 < \ldots < b_{2p-1} < b_{2p} < b_{2p-2} < \ldots < b_2,$$

then

$$b_{2p+1} = b^{b_{2p}} < b^{b_{2p-1}} = b_{2p},$$
$$b_{2p-1} = b^{b_{2p-2}} < b^{b_{2p}} = b_{2p+1},$$
$$b_{2p+2} = b^{b_{2p+1}} < b^{b_{2p-1}} = b_{2p}$$

and so it follows, by induction, that b_{2n+1} is monotonic increasing and bounded above, and b_{2n} is monotonic decreasing and bounded below, so that both b_{2n} and b_{2n+1} are convergent. To determine these limits we must examine the roots of the equation $b^{b^x} = x$.

If β is the unique number in $(0, 1)$ such that $\beta^{1/\beta} = b$ then $b^\beta = \beta$ and so $x = \beta$ satisfies $b^{b^x} = x$. Let $b^x = y$ then $b^y = x$ and so $x^x = y^y$. As x increases from 0 to 1, x^x steadily decreases from 1 to $e^{-1/e}$ at $x = 1/e$, and then steadily increases to 1. Thus there is a $(1, 1)$ correspondence between x in $0 < x < 1/e$ and y in $1/e < y < 1$, such that $x^x = y^y$; if ξ, η are a corresponding pair and $b = \xi^{1/\eta}$ then $b^{b^x} = x$ has the three solutions $x = \xi$, $x = \beta$, $x = \eta$, where $b^\xi = \eta$, $b^\eta = \xi$, $b^\beta = \beta$.

To solve the equation $x^x = y^y$, set $y = x^c$, $0 < c < 1$, then $x^x = x^{cy}$ so that $x = cy = cx^c$, whence

$$x = c^{1/(1-c)}, \quad y = c^{c/(1-c)};$$

writing $c = 1 - 1/k$, we have, for $k > 1$,

$$x = \left(1 - \frac{1}{k}\right)^k, \quad y = \left(1 - \frac{1}{k}\right)^{k-1}, \quad b = y^{1/x}.$$

A simple calculation shows that the value of $(y/bk)\dfrac{db}{dk}$ is

$$\frac{1}{k(1-k)} - \left\{\log\left(1 - \frac{1}{k}\right)\right\}^2$$
$$= \int_0^1 \frac{du}{(u+k-1)^2} - \left\{\int_0^1 \frac{du}{u+k-1}\right\}^2 > 0$$

so that b steadily increases and tends to $1/e^e$ as $k \longrightarrow \infty$. It follows that if $1 \geq b \geq 1/e^e$ the equation $b^{b^x} = x$ has the single solution $x = \beta$, where $\beta^{1/\beta} = b$; since the limits of b_{2n} and of b_{2n+1} both satisfy $b^{b^x} = x$, therefore $b_n \longrightarrow \beta$, whether n be odd or even, when $1 \geq b \geq 1/e^e$.

In the case $0 < b < 1/e^e$, let ξ, β, η with $0 < \xi < \beta < \eta < 1$, be the roots of the equation $b^{b^x} = x$; then (since $b^{b^x} \longrightarrow b > 0$, as $x \longrightarrow 0$) we have (i) $b^{b^x} > x$ for $0 < x < \xi$ and for $\beta < x < \eta$; (ii) $b^{b^x} < x$ for $\xi < x < \beta$ and for $\eta < x < 1$. As $b < \beta < b^b$, and if for some p, $b_{2p-1} < \beta < b_{2p}$ then $b_{2p+1} = b^{b_{2p}} < b^\beta = \beta$ and $b_{2p+2} = b^{b_{2p+1}} > b^\beta = \beta$ therefore $b_{2p+1} < \beta < b_{2n}$ for all n. Since however $b_{2n-1} < b_{2n+1} = b^{b^{b_{2n-1}}}$ therefore $0 < b_{2n-1} < \xi$ for all n and similarly $\eta < b_{2n} < 1$ for all n, from which it follows that $b_{2n+1} \longrightarrow \xi$, $b_{2n} \longrightarrow \eta$ (since lim b_{2n} and lim b_{2n+1} are roots of the equation $b^{b^x} = x$.

Another treatment of the equation $b^{b^x} = x$ was given by Mr. A. G. Vosper in note 2517.

Defining, as in the previous note, $L^0x = x$, $L^{n+1}(x) = \log_b L^n(x)$, we see that in order that L^nx should exist for all n, when $0 < b < 1$, we require that $L^nx > b$ for all n, and therefore $b_{2n-1} < x < b_{2n}$ for all n (since $L^{2n}x > b$ implies $x > b_{2n+1}$ and $L^{2n-1}x > b$ implies $x < b_{2n}$). Hence, if $1/e^e \leq b < 1$, then L^nx exists for all n only when $x = \beta$ and then $L^nx = \beta$ for all n.

If $0 < b < 1/e^e$ then L^nx exists for all n only in the interval $\xi \leq x \leq \eta$; to find the limit of L^nx we distinguish the cases

$$\xi < x < \beta, \quad \beta < x < \eta.$$

If $\xi < x < \beta$ then $x > b^{b^x}$, so that $L^2x > x$ and therefore

$$\xi < L^{2n}x < L^{2n+2}x < \beta;$$

but $\beta = \log_b \beta < \log_b x < \log_b \xi = \eta$, (and from $\beta < y < \eta$ follows $L^2y < y$) so that $\beta < L^{2n+1}x < L^{2n-1}x < \eta$.

Thus $L^{2n}x$ and $L^{2n+1}x$ are both convergent, and since β is the only root of $L^2x = x$ between ξ and η we conclude that $L^n x \longrightarrow \beta$, whether n is odd or even.

Similarly, in the case $\beta < x < \eta$, $L^n x \longrightarrow \beta$. If $x = \beta$ then $L^n x = \beta$ for all n. If $x = \xi$, $L^{2n}x = \xi$ and $L^{2n+1}x = \eta$ and if $x = \eta$ then $L^{2n}x = \eta$, and $L^{2n+1}x = \xi$ for all x.

We observe next that the behaviour of the function $b_n(x)$, with $b_0(x) = x$, $b_{n+1}(x) = b^{b_n(x)}$, closely follows that of b_n itself. For $b > e^{1/e}$ and any x, positive or negative, b^x is positive and $b_n(x)$ steadily increases to infinity. If $b = 1$, of course $b_n(x) = 1$ for all x. If $1 < b \leq e^{1/e}$ we distinguish the cases

(1) $x > B$, then $b_n(x)$ steadily increases to infinity.

(2) $x = B$, then $b_n(x) = B$.

(3) $\beta < x < B$ then $b_n(x)$ steadily decreases to β.

(4) $x = \beta$, then $b_n(x) = \beta$.

(5) $0 \leq x < \beta$, then $b_n(x)$ steadily increases to β.

(6) $x < 0$, then $b^x < 1 < \beta$ and so $b_n(x) \longrightarrow \beta$ as in (5).

Finally, in the case $0 < b < 1$ we have

(1) $x > 0$ then $b^x < 1$, and $b_{2n+1}(x)$ increases and $b_{2n}(x)$ decreases and $\lim b_n(x) = \lim b_n$.

(2) $x < 0$, then $0 < b^{b^x} < 1$ and $\lim b_n(x) = \lim b_n$ as in (1).

(3) $x = 0$ then $b_{n+1}(x) = b_n$.

We come now to the system of equations

$$b^x = x_1, b^{x_1} = x_2, \ldots, b^{x_n} = x, b > 0,$$

which are equivalent to

$$b_{n+1}(x) = x, x_1 = b_1(x), x_2 = b_2(x), \ldots, x_n = b_n(x).$$

Since $b^x > 0$ whether x is positive or not, it suffices to consider the case $x > 0$. If $b = 1$, the only solution is $x = 1$. If $b > 1$ and $b > x^{1/x}$ then $b^x > x$ and so $b_n(x) > x$ for all; and if $b > 1$ and $b < x^{1/x}$ then $b_n(x) < x$ for all x. Thus for $b > 1$ the only solutions of the equation $b_{n+1}(x) = x$ are $x = \beta$, B where $\beta^{1/\beta} = B^{1/B} = b$, and these solutions exist only if $b < e^{1/e}$ (and coincide if $b = e^{1/e}$). The interesting case is $0 < b < 1$. We shall show that if n is even the equation

$$b_{n+1}(x) = x$$

has just the single solution $x = \beta$ (where $\beta^{1/\beta} = b$) and if n is odd the equation has the same solutions as the equation $b_2(x) = x$, considered above.

For if

$$b_2(x) = x$$

then

$$b_4(x) = b_2(x) = x,\ b_6(x) = b_2(x) = x,$$

and so on, and of course if $b_1(x) = x$ then

$$b_2(x) = b_1(x) = x,\ b_3(x) = b_1(x) = x,$$

and so on. And furthermore if $b_1(x) > x$ then $b_3(x) > x$ and hence $b_5(x) > x$, and so on, and similarly if $b_1(x) < x$ then $b_3(x) < x$, $b_5(x) < x, \ldots$ in turn. Finally we observe that $b_4(x) \gtrless x$ according as $b_2(x) \gtrless x$ and if $b_2(x) > x$ and $b_n(x) > x$ then $b_{n+2}(x) > 0$ and if $b_2(x) < x$ and $b_n(x) < x$ then $b_{n+2}(x) < x$, which completes the proof.

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2805. Ill-conditioned linear equations

The equations set out below are given in a paper by L. Fox[1] (1948), where they are ascribed to T. S. Wilson. The same equations are given by R. V. Southwell[2] (1949), and by F. S. Shaw[3] (1953). They are also to be found in the recent book by L. Fox[4] (1957). The point made in all these references is that a solution by a "Southwell relaxation" technique is, for practical purposes, impossible. The table shows the values of the variables, which, on substitution into the equations, produce the corresponding, exact, residuals.

$$\begin{aligned} 5x_1 + 7x_2 + 6x_3 + 5x_4 - 23 &= 0 \\ 7x_1 + 10x_2 + 8x_3 + 7x_4 - 32 &= 0 \\ 6x_1 + 8x_2 + 10x_3 + 9x_4 - 33 &= 0 \\ 5x_1 + 7x_2 + 9x_3 + 10x_4 - 31 &= 0 \end{aligned}$$

Variables				Residuals			
x_1	x_2	x_3	x_4				
14·6	−7·2	−2·5	3·1	0·1	−0·1	−0·1	0·1
2·36	0·18	0·65	1·21	0·01	−0·01	−0·01	0·01
1·136	0·918	0·965	1·021	0·001	−0·001	−0·001	0·001
1	1	1	1	0	0	0	0

It might be supposed that ill-conditioned sets of equations, with symmetric matrix, and unit determinant, such as the above, are somewhat difficult to construct; however, this is not so, as trial shows. The following set of equations, with coefficients of the same

order as those above, yields residuals ten times smaller, for values of the variables which are still up to twenty per cent in error. The interesting problem, unsolved as far as the writer is aware, is just how small the residuals may be made while the error in the solution persists at this level.

$$\begin{aligned} 6x_1 + 7x_2 + 10x_3 + 10x_4 - 33 &= 0 \\ 7x_1 + 8x_2 + 13x_3 + 11x_4 - 39 &= 0 \\ 10x_1 + 13x_2 + 5x_3 + 19x_4 - 47 &= 0 \\ 10x_1 + 11x_2 + 19x_3 + 6x_4 - 46 &= 0 \end{aligned}$$

Variables				Residuals			
x_1	x_2	x_3	x_4				
222·3	−172·1	−17·4	7·8	0·1	−0·1	−0·1	0·1
23·13	−16·31	−0·84	1·68	0·01	−0·01	−0·01	0·01
3·213	−0·731	0·816	1·068	0·001	−0·001	−0·001	0·001
1·2213	0·8269	0·9816	1·0068	0·0001	−0·0001	−0·0001	0·0001
1	1	1	1	0	0	0	0

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2. R. V. Southwell. Engineering Structures. Supplement to "Research", 1949.

3. F. S. Shaw, An introduction to relaxation methods. Dover Publications, 1953.

4. L. Fox, The numerical solution of two-point boundary problems in ordinary differential equations. Oxford University Press, 1957.

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2806. On Note 2706

The Theorem of Note 2706 (1957, p. 206) makes a pleasant exercise in areal coordinates. Let ABC be the triangle of reference, and let D, D'; E, E'; F, F' have coordinates $(0, d, 1)$, $(0, d', 1)$; $(1, 0, e)$; $(1, 0, e')$; $(f, 1, 0)$, $(f', 1, 0)$. Then BE is $z = ex$, CF' is $x = f'y$ and so AX is $z = ef'y$ and similarly BY is $x = fd'z$, CZ is $y = de'x$ and these are concurrent if $def\, d'e'f' = 1$.

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2807. A note on Laplace transforms

Sufficient conditions that $f(t) \supset \phi(p)$ and $\xi(x, t) \supset \phi_1(p)h(p)e^{-xh(p)}$ imply that $\int_0^\infty \xi(x, t)f(x)\,dx \supset \phi_1(p)\phi[h(p)]$ were stated by N. W. McLachlan in the Mathematical Gazette, Vol. 30 (1946) p. 85. $\left(f(t) \supset \phi(p)\right.$ denotes the relation $\left.\phi(p) = p\int_0^\infty e^{-pt}f(t)\,dt.\right)$ An interesting special case is obtained by setting $\phi_1(p) = [h(p)]^a$ and $\phi(p) = p^{c-a}$, with $c < a + 1$ since $\phi(p)$ is to be a Laplace transform. Then

$$\frac{1}{\Gamma(a-c+1)}\int_0^\infty \frac{\xi(x, t)}{x^{c-a}}\,dx \supset [h(p)]^c$$

where

$$\xi(x, t) \supset [h(p)]^{a+1}e^{-xh(p)}.$$

The parameter a allows some freedom of choice in selecting the function $\xi(x, t)$.

A simple example shows that there actually are functions $h(p)$ and $\xi(x, t)$ satisfying the relations. If $h(p) = p^{-1}$, then $\xi(x, t) \supset p^{-(a+1)}e^{-x/p}$. Let $a = -1/2$ so that $\xi(x, t) = (\pi x)^{-1/2}\sin 2\sqrt{xt}$ can be obtained from tables of transforms. Substituting into the integral, evaluating the integral, and simplifying the result, one obtains

$$\frac{1}{\Gamma(1/2-c)}\int_0^\infty \frac{\sin 2\sqrt{xt}\,dx}{\sqrt{\pi}x^{1+c}} = \frac{2^{2c+1}t^c\Gamma(-2c)\sin(-\pi c)}{\sqrt{\pi}\Gamma(1/2-c)}$$

$$= \frac{t^c}{\Gamma(1+c)} \supset \frac{1}{p^c} = (p^{-1})^c,$$

which is the desired result, and which is valid for $-1/2 < c < 1/2$.

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2808. A property of a plane convex region

Let D be a plane convex region whose boundary B consists of a finite number of rectifiable arcs. Then there exists at least one straight line which bisects both B and the area of D.

For let B be oriented in a definite sense and parametrized by means of its arc-length. Through any point P_1 of B there passes a unique line which bisects B and meets B in Q_2. Through Q_2 there passes a unique line which bisects the area of D and meets B in Q_1. Through Q_1 there passes a unique line which bisects B and meets B in P_2.

Now either P_1 and Q_1 coincide and the theorem is proved, or P_1Q_2 and P_2Q_1 intersect in an interior point of D. Let the arc-length $P_1Q_1 = f(P_1)$. Then f is a one-valued continuous function of the position of P_1 which may be regarded as positive if P_1Q_1 has the orientation of B and negative otherwise. Clearly $f(P_1)$ and $f(P_2)$ have opposite signs. Hence $f(P) = 0$ for at least one point P on the arc P_1P_2 and the result follows.

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2809. A note on rocket staging

In an article in "The New Scientist" Professor Baxter[1] points out that the advantages obtained by "staging" a rocket are limited. He comments that the gain in later stages grows progressively smaller. In fact one can go further than this and show that the effective mass ratio of a multi-stage rocket has a definite maximum as a function of the number of stages.

The Effective Mass Ratio

Let us suppose that a rocket of initial mass M is divided into 'n' stages, each having the same mass ratio 'r', the final stage—the payload—having unit mass, and in each stage let the structural mass be a fraction 's' of its initial mass. Since the mass ratio of any stage is the ratio of the initial to the final mass of that stage we have for the tth stage

$$r = m_t/(m_{t+1} + sm_t),$$

where m_t is the mass of the tth stage.

Hence $m_t = rm_{t+1}/(1 - sr)$ and by induction

$$M = r^n/(1 - sr)^n.$$

The effective mass ratio R is given by r^n and therefore

$$R = M/(1 + sM^{1/n})^n.$$

The Condition for Maximum R

It is sufficient to consider stationary values of the function

$$y = (1 + sM^{1/n})^n.$$

Writing $1/n = \tau$, we have that

$$y^\tau = 1 + sM^\tau$$

whence

$$\tau y^{\tau-1}\frac{dy}{d\tau} + y^\tau \log y = sM^\tau \log M.$$

[1] Baxter, A. D., "The New Scientist", 9th January, 1958.

Stationary values of y occur when

$$sM^r \log M = y^r \log y,$$

or,

$$sM^r \log M^r = (1 + sM^r) \log (1 + sM^r).$$

If we write $M^r = M^{1/n} = x$, this takes the form

$$x^{sx} = (1 + sx)^{(1+sx)}$$

and the problem is essentially to determine x as a function of s. Thus

$$x = (1 + sx)^{1/sx} \,.\, (1 + sx)$$

and evidently the limit of x as $s \to 0$ is 'e', so that even in this case there is an upper limit to the number of useful stages when

$$n = \log M/\log e.$$

In practice sx is less than unity and we can employ the binomial expansion of the term $(1 + sx)^{1/sx}$. It is however easier to use the Taylor expansion since the re-grouping of the binomial series is rather laborious. Thus, let

$$w = (1 + \alpha)^{1/\alpha}$$

whence

$$\alpha w' + w \log w = w/(1 + \alpha),$$

dashes denoting differentiation with respect to α, and by a second differentiation

$$\alpha w'' + w'\{2 - \alpha w'/w\} = -w/(1 + \alpha)^2.$$

Since $\lim\limits_{\alpha\to 0} w = e$, we find that

$$w'(0) = -e/2.$$

Similarly

$$\alpha w''' + 3w'' - \frac{\alpha d}{d\alpha}\left(\frac{w'^2}{w}\right) - \frac{w'^2}{w} + \frac{w'}{(1 + \alpha)^2} = \frac{2w}{(1 + \alpha)^3}$$

and

$$w''(0) = +\frac{11}{12}\,e.$$

A further differentiation shows that

$$w'''(0) = -\frac{17}{8}\,e.$$

Thus

$$w = e\left\{1 - \frac{1}{2}\,\alpha + \frac{11}{12}\frac{\alpha^2}{2!} - \frac{17}{8}\frac{\alpha^3}{3!} + \dots\right\}$$

and therefore

$$x = e\left\{1 - \frac{sx}{2} + \frac{11}{24}(sx)^2 - \frac{17}{48}(sx)^3 + \ldots\right\}(1 + sx) \qquad (1)$$

$$= e\left\{1 + \frac{sx}{2} - \frac{1}{24}(sx)^2 + \frac{5}{48}(sx)^3 - \ldots\right\}. \qquad (2)$$

Hence as a first approximation

$$x = e\Big/\left(1 - \frac{es}{2}\right).$$

In practice s is 20% or less, so that $x \sim 3{\cdot}7$. Further the coefficients in the series in equation (1) are dominated by the right hand members of the differential relations above and therefore the coefficient of $(sx)^r$ is of the same order as $1/r$. Hence for the series in (2) the coefficients are alternating in sign and ultimately monotonic decreasing, so that the error in dropping the term in $(sx)^2$ and higher powers will be of the order of the first term dropped. Thus the error in x is given by

$$\delta x \approx \frac{e}{24}(sx)^2$$

$$\approx 0{\cdot}063.$$

Further, since $x = M^{1/n}$

$$\delta n = -\frac{n^2}{x}\frac{\delta x}{\log e\, M}$$

where $M \sim 1000$ and n is not more than 10. Thus

$$\delta n \approx -0{\cdot}2$$

and the first approximation is sufficient to determine the integer n.

Hence for optimum performance the maximum number of stages is given by

$$n = \left[\log M\Big/\left(\log e - \log\left(1 - \frac{es}{2}\right)\right)\right],$$

taking the value 5 for $M = 1000$, $s = 0{\cdot}2$ and varying very slowly as a function of M and s. In practice it will not normally be worthwhile to use the optimum number of stages since R varies slowly near its maximum as a function of n, and the American use of four stages for a satellite rocket is probably the best compromise between a technical maximum and technological difficulties.

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2810. On approximations

Approximations of the type $29\sqrt{14} + 4\sqrt{15} \doteqdot 124$ are tolerably familiar. The general formula is

(a) $(2n+1)\sqrt{(n+1)} - (2n-1)\sqrt{(n-1)} \doteqdot 4\sqrt{n} + 1/(32\,n^{7/2})$;

and, for cube roots,

(b) $(9n+5)\sqrt[3]{(n+1)} - (9n-5)\sqrt[3]{(n-1)} \doteqdot 16\sqrt[3]{n} + 32/(243n^{11/3})$.

These are special cases of the general result

(c) $$(n+\alpha)(n+1)^p - (n-\alpha)(n-1)^p = \frac{4}{3}(p+1)n^p - \frac{4}{3}\binom{p+1}{5}n^{p-4} + \ldots$$

with

$$\alpha = (2-p)/3.$$

For the first terms in the expansion of the left-hand side of (c) are

$$2(\alpha+p)n^p + 2\left\{\alpha\binom{p}{2} + \binom{p}{3}\right\}n^{p-2} + 2\left\{\alpha\binom{p}{4} + \binom{p}{5}\right\}n^{p-4},$$

and the second of these vanishes if

$$\alpha = -\binom{p}{3}\Big/\binom{p}{2} = -\frac{p-2}{3}.$$

The approximations are very close for moderately large n; e.g. in the example given $29\sqrt{14} + 3\sqrt{15} = 123{\cdot}9999976$.

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2811. A curious triangle

In Note 2744, vol. XLII, no. 339, Feb. 1958, p. 40, A. K. Rajagopal has mentioned a few properties of a special triangle which is similar to the triangle having as sides its medians.

Neuberg has called such a triangle "automedian", taking into account the following property: a triangle $A'B'C'$ being constructed with its sides parallel and proportional to the medians of a triangle ABC, the medians of $A'B'C'$ will then also be parallel and proportional to the sides of ABC. Each of the triangles ABC, $A'B'C'$ is therefore called "median" of the other. But the two triangles $A'B'C'$ and ACB may be similar and then ABC is "automedian".

A great number of interesting properties of this special triangle have been given (Artzt, *Zeitschrift fur mathematischen und naturwissenschaflige Unterricht*, 1887, p. 198, Stegeman, *Ibid.*, 1888,

p. 32, Neuberg, *Mathesis*, 1889, p. 208, 1901, p. 280, Cristescu, *Ibid.*, 1902, p. 205, Deprez, *Ibid.*, 1903, pp. 196, 226, 245, Goormaghtigh, *Ibid.*, 1913, p. 12, 1914, p. 182, Neuberg, *Mémoires de la Société Royale des Sciences de Liège*, 1924, Goormaghtigh, *Ibid.*, 1925.

R. GOORMAGHTIGH

2812. Approximate length of arc of an ellipse

A very close approximation has been given by C. M. Segedin in Note 2722, vol. XLI, No. 338, Dec. 1957, p. 275: I have mentioned the same in the American Mathematical Monthly, 1930, p. 441 and in Mathesis, 1930, p. 250.

R. GOORMAGHTIGH

2813. A historical note on Note 2726

Leech's problem of constructing a two-way infinite sequence of 0's, 1's, and 2's such that no two consecutive blocks of symbols are identical was, as far as I know, first considered by Axel Thue. There is a long paper by him 'Uber die gegenseitige Lage gleicher Teile gewisser Zeichenreihen' (Skrifter utgit av Videnskapsselskapet i Kristiania, I. Mat-Nat. Klasse I Bind (1912) pp. 1–67) in which he states the general problem of constructing two-way infinite sequences on n symbols such that two identical sub-sequences are always separated by at least $n - 2$ symbols. (He points out that his remarks apply also to the construction of one-way infinite and arbitrarily long finite closed (i.e. cyclic instead of linear order) sequences.) His method of constructing such 'irreducible' sequences is the same as Leech's viz. by iterating the process of substituting fixed blocks of symbols for the basic symbols. He starts with the case $n = 2$; here an irreducible sequence is simply one in which there is no pair of identical (but not coincident) sub-sequences which overlap. He shows that such a sequence can be obtained by starting with the sequence 00 and iterating the process of replacing 0 by 0110 and 1 by 1001. His solutions for the case $n = 3$ are shorter than Leech's (although they do not possess the symmetry of these); one of them is to start with 01201 and repeatedly replace 0 by 01201, 1 by 020121, 2 by 0212021. He points out that every two-way infinite irreducible sequence on 3 letters can be built up from blocks of the forms 01, 02, 012, 021, 0121, 0212 and proceeds to consider in detail the various possible kinds of irreducible sequences which can be formed using only four of these six blocks. Finally he shows how to construct irreducible sequences on n symbols for $n > 3$ by starting with an irreducible sequence on $0, 1, 2$ and replacing $0, 1, 2$ by suitably chosen finite sequences on the n symbols.

J. C. SHEPHERDSON

2814. Transport problems

The problem posed in Note 2609 suggests a number of interesting exercises in probability theory. A few examples follow. Problems 1 to 3 are relatively simple, but they prepare the way for the more diffcult problem 4.

Problem 1. I arrive at a bus stop where I have a choice of n independent bus systems, on each of which buses run at intervals T. How long may I expect to wait for a bus?

Taking the instant of my arrival as the origin of time t, the probability that the next bus of a specified system will arrive later than time t is $(1 - t/T)$. Thus the probability that the waiting time exceeds t is

$$Q_n(t) = (1 - t/T)^n, \quad 0 \leqslant t \leqslant T.$$

Hence the probability that the waiting time lies between t and $t + dt$ is

$$P_n(t)dt = -Q'_n(t)dt = nT^{-1}(1 - t/T)^{n-1}dt,$$

and the expectation of the waiting time is

$$\bar{t} = \int_0^T tP_n(t)\,dt = \int_0^T Q_n(t)\,dt = T/(n + 1).$$

Problem 2. This is the same as Problem 1, but with the additional information that no bus has passed between time $-u$ and the present.

With an obvious modification in the above notation,

$$Q_n(t, u) = \{1 - t/(T - u)\}^n, \quad 0 \leqslant t \leqslant T - u,$$

$$P_n(t, u)\,dt = -(d/dt)Q_n(t, u)\,dt,$$

$$\bar{t} = (T - u)/(n + 1).$$

Problem 3. This is the same as Problem 2, but with the additional information that a bus passed at the instant $-u$.

The solution is the same as in Problem 2, but with n replaced by $n - 1$, since a bus from each of the other $n - 1$ systems will arrive before time $T - u$. If $n = 1$ there is only one possible waiting time $T - u$.

Problem 4. This is the same as Problem 1, but with the following additional information. On my arrival, I find r other passengers already waiting at the bus stop. Passengers are known to arrive at the bus stop independently of one another in a random stream at an average rate a per unit time, that is to say, the probability that a passenger will arrive during any infinitesimal interval dt is $a\,dt$. We suppose that the latest bus picked up all passengers then waiting.

Regarding r as variable for the moment, we consider the probability of the truth of three statements made at the time of my arrival:

(A) The latest bus passed between times $-u$ and $-u + du$.

(B) There are r other passengers already waiting.

(C) My waiting time will lie between t and $t + dt$.

Disregarding (B) and (C), the probability that (A) is true is $P_n(u)\,du$, $0 \leqslant u \leqslant T$, by analogy with Problem 1.

The conditional probability that (B) is true, on the assumption that (A) is true, is given by Poisson's distribution:

$$p_r(u) = (au)^r e^{-au}/r!, \qquad r = 0, 1, 2, \ldots$$

Hence the probability that (A), (B) are both true is $p_r(u)P_n(u)\,du$. If this expression is summed for $r = 0, 1, 2, \ldots$ and integrated from $u = 0$ to $u = T$, we correctly obtain unity. However, if we disregard all values of r except that specified in the problem, that is to say if (B) is known to be true, the distribution must be renormalized by introducing a factor K_r such that

$$K_r \int_0^T p_r(u) P_n(u)\,du = 1.$$

Thus we find that the conditional probability that (A) is true, on the assumption that (B) is true, is

$$K_r p_r(u) P_n(u)\,du.$$

Now the conditional probability that (C) is true, on the assumption that (A) is true, is

$$P_{n-1}(t, u)\,dt, \quad 0 \leqslant t \leqslant T - u, \quad n > 1,$$

using Problem 3. Hence the probability that (A), (C) are both true when r has been assigned is

$$K_r p_r(u) P_n(u) P_{n-1}(t, u)\,du\,dt = nT^{-1} K_r p_r(u) P_{n-1}(u + t)\,du\,dt.$$

Finally, integrating over all permissible values of u, the probability that (C) is true when r has been assigned is

$$F_n(t, r)\,dt = nT^{-1} K_r\,dt \int_0^{T-t} p_r(u) P_{n-1}(u + t)\,du,$$

and this is the required distribution of the waiting time t when $n > 1$. The expectation of the waiting time is

$$\bar{t} = \int_0^T t F_n(t, r)\,dt,$$

which, on changing the order of integration and carrying out the

integration with respect to t, becomes

$$l = K_r \int_0^T p_r(u)(1 - u/T)^n \, du.$$

When there is only one bus system, we find by similar reasoning that

$$F_1(t, r) \, dt = K_r T^{-1} p_r(T - t) \, dt,$$

while the expression given above for l remains valid with $n = 1$.

E. J. BURR

2815. A theorem of Besicovitch

We deal with real-valued functions defined on the interval $(0, 1)$. Given a sub-interval (a, b), the *smoothing operator* T relative to this sub-interval replaces a function f by the function $T(f)$ which is identical with f outside of the sub-interval and which has for its graph the line segment from $(a, f(a))$ to $(b, f(b))$ on the sub-interval. The following result was communicated to the author by Professor Besicovitch. We indicate a proof somewhat simpler than the original one.

THEOREM. Let $\{T_n\}$ be an arbitrary sequence of smoothing operators and let f be continuous on $(0, 1)$. Set $f_0 = f, f_n = T_n(f_{n-1})$. Then the sequence $\{f_n\}$ converges uniformly.

We first note some elementary properties of a smoothing operator T:

(1) T is linear, $T(af + bg) = aT(f) + bT(g)$.

(2) T sends convex functions to convex functions and concave functions to concave functions.

(3) $T(f)$ is continuous if f is continuous.

We need one further property which we express by means of the uniform norm: $\|f\| = \sup |f|$.

(4) $\|Tf\| \leqslant \|f\|$.

Proof of Theorem. First suppose $f = f_0$ is a concave function. Then each f_n is concave and clearly $f_0(x) \leqslant f_1(x) \leqslant f_2(x) \leqslant \ldots$ for each x. That is, the sequence $\{f_n\}$ is an increasing sequence of concave functions. The sequence is also bounded since $\|f_n\| \leqslant \|f\|$. Consequently the sequence converges pointwise to a concave, hence continuous, function. But this already tells us that the convergence is uniform by a well-known theorem since $\lim f_n$ is a continuous limit of an increasing sequence of continuous functions. We have proved the theorem in case f is concave, hence also in case f is convex.

In the general case, given f and given $\varepsilon > 0$, we may find a function $g = g^1 + \ldots + g^r$ such that $\|f - g\| < \varepsilon$ with each g^i

concave or convex. For example we may use the Weierstrass approximation theorem and have each g^i a monomial of the form ax^m. For each i the sequence $\{g_n{}^i\}$ converges uniformly, hence there exists an N such that $\|g_m - g_n\| < \varepsilon$ whenever $m, n \geqslant N$. But $\|f_m - g_m\| \leqslant \|f - g\| < \varepsilon$, $\|f_n - g_n\| < \varepsilon$, so we conclude that $\|f_m - f_n\| < 3\varepsilon$ for all $m, n \geqslant N$. This establishes the uniform convergence of $\{f_n\}$.

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2816. A new proof of the compositeness of F_5

$$\begin{aligned} F_5 &= 2^{32} + 1 \\ &= 15 \,.\, 2^{28} + 2^{28} + 1 \\ &= 15 \,.\, (2^7)^4 + (5^3 + 3)(2^7)^3 + 1 \\ &= 3 \,.\, (2^7)^3[5 \,.\, 2^7 + 1] + (5 \,.\, 2^7)^3 + 1 \end{aligned}$$

Hence F_5 is divisible by $5 \,.\, 2^7 + 1 = 641$.

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2817. A numerical method of solving first order differential equations

The method here given is a simple modification of Methods I and II in W. E. Milne's Numerical Solution of Differential Equations. It gives a higher order of accuracy in return for a negligible amount of extra work. Since numerical methods are, regrettably, hardly touched on in school work, Milne's book is probably not widely available and the following account is therefore made self-contained.

The problem is to tabulate that solution of

$$\frac{dy}{dx} = f(x, y) \qquad (1)$$

for which $y = y_0$ when $x = x_0$, the values of y, corresponding to the values x_0, $x_0 + h$, $x_0 + 2h$, . . . of x, being found. Denote these values by $y_0, y_1, y_2, \ldots$.

In order to start the solution, y_1 must first be found, e.g. from the Taylor series

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \cdots$$

the derivatives being calculated from the differential equation itself.

Now

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2!}y_n'' + \frac{h^3}{3!}y_n''' + \frac{h^4}{4!}y_n^{\mathrm{IV}} + \frac{h^5}{5!}y_n^{\mathrm{V}} + \dots$$

and

$$y_{n-1} = y_n - hy_n' + \frac{h^2}{2!}y_n'' - \frac{h^3}{3!}y_n''' + \frac{h^4}{4!}y_n^{\mathrm{IV}} - \frac{h^5}{5!}y_n^{\mathrm{V}} + \dots$$

and hence

$$y_{n+1} - y_{n-1} = 2hy_n' + \frac{h^3}{3}y_n''' + \frac{h^5}{60}y_n^{\mathrm{V}} + \dots \tag{2}$$

It follows that

$$y_{n+1} \simeq y_{n-1} + 2hy_n' \tag{3}$$

Milne's Method I consists of repeatedly applying this formula. If y_n is known, y_n' is calculated from (1), and then y_{n+1} from (3).

Milne's Method II uses a rather more accurate formula than (3):

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2!}y_n'' + \frac{h^3}{3!}y_n''' + \frac{h^4}{4!}y_n^{\mathrm{IV}} + \dots$$

$$y_{n+1}' = y_n' + hy_n'' + \frac{h^2}{2!}y_n''' + \frac{h^3}{3!}y_n^{\mathrm{IV}} + \dots$$

Hence

$$y_{n+1} - \frac{h}{2}y_{n+1}' = y_n + \frac{h}{2}y_n' - \frac{h^3}{12}y_n''' - \frac{h^4}{12}y_n^{\mathrm{IV}} + \dots \tag{4}$$

so that

$$y_{n+1} \simeq y_n + \frac{h}{2}(y_{n+1}' + y_n') \tag{5}$$

Formula (3) is used as a "predictor" for the value of y_{n+1}. This value is then inserted in (1) and a value for y_{n+1}' obtained. Inserting this in (5) gives a "corrected" value for y_{n+1}. A new value for y_{n+1}' is then calculated from the corrected value of y_{n+1}, and y_{n+1} recalculated again from (5) and the process repeated until no further change takes place in the calculated value of y_{n+1}. Milne's Method II takes this value as the true value of y_{n+1}.

Now, inspection of (2) and (4) shows that the principal part of the errors in (3) and (5) are $\frac{h^3}{3}y_n'''$ and $\frac{h^3}{12}y_n'''$ in defect and excess respectively; and hence a better value than that obtained by Milne's Method II can be found by interpolating between the values given by (3) and (5) in the ratio 4 : 1. This is equivalent to using the formula

$$y_{n+1} = 0{\cdot}8\,y_n + 0{\cdot}2y_{n-1} + 0{\cdot}4h(y_{n+1}' + 2y_n') \tag{6}$$

and the first neglected term in this expansion is $-\frac{h^4}{15}y_n^{\text{IV}}$. The actual error in y_{n+1} depends, however, on the form of the function $f(x, y)$. For the error in the "predicted" value of y_{n+1} is approximately $\frac{h^3}{3}y_n'''$, and y'_{n+1} is found in the first instance as $f(x_{n+1}, y_{n+1})$ using the predicted value. If the form of f is such that the error in this value for y'_{n+1} is also $O(h^3)$, then the error in y_{n+1} obtained from a first application of (5) is $O(h^4)$. Recalculating y'_{n+1} from this new value for y_{n+1} should give a better value for y'_{n+1}, and if this value is in error by $O(h^4)$ then the final error in y_{n+1} due to error in y'_{n+1} will be $O(h^5)$.

Formula (6) above provides a useful check on the accuracy of the arithmetic, and when the final value for y_{n+1} has been obtained by the interpolation method it should be recalculated from (6).

Illustrative Example

Tabulate that solution of $\frac{dy}{dx} = xy$ for which $y = 1$ when $x = 0$. (This solution is $y = e^{x^2/2}$, values of which are given below for comparison purposes.)

We take $h = 0{\cdot}1$, and first find y_1.

Now, $y_0 = 1$, $y_0' = 0$.

$$y'' = xy' + y = (x^2 + 1)y \qquad \therefore\ y_0'' = 1$$

$$y''' = (x^2 + 1)y' + 2xy = (x^3 + 3x)y \qquad \therefore\ y_0''' = 0$$

$$y^{\text{IV}} = (x^3 + 3x)y' + (3x^2 + 3)y \qquad \therefore\ y_0^{\text{IV}} = 3$$

Hence,

$$y_1 \simeq 1 + \frac{0{\cdot}01}{2} \times 1 + \frac{0{\cdot}0001}{24} \times 3$$

$$\simeq 1{\cdot}00501$$

Since in this case $\frac{h^4}{15}y_0^{\text{IV}} = 0{\cdot}00002$, we cannot expect accuracy to five decimal places, but four-place accuracy should be obtainable, at least to begin with.

It is advantageous to arrange the work in columns with suitable headings. In the table below, column 3 gives the "predicted" value of y_n, column 4 the value of y_n' obtained from the value in column 3, column 5 the "corrected" value of y_n using column 4 in the first instance, column 6 the interpolated value for y_n and column 7 the value of y_n' obtained from the value in column 6. If the value in column 7 differs from that in column 4 by an amount which would affect the "corrected" value in column 5 if it were used for calculating

this value, then columns 5, 6 and 7 are re-worked using the column 7 value for y_n'. If using the new column 7 value would alter column 5, then columns 5, 6 and 7 are again re-worked and the process continued until no further change takes place. The discarded values in columns 6 and 7 are bracketed as a safeguard against their being used again.

1	2	3	4	5	6	7	
n	x_n	$y_{n-2} +$ $0{\cdot}2\, y'_{n-1}$	$y_n'(1)$	$y_{n-1} + 0{\cdot}05(y'_{n-1} + y_n')$	y_n	y_n'	$e^{x_n^2/2}$
0	0				1	0	1
1	0·1				1·0050	0·1005	1·0050
2	0·2	1·0201	0·2040	1·0202	1·0202	0·2040	1·0202
3	0·3	1·0458	0·3137	1·0461	1·0460	0·3138	1·0460
4	0·4	1·0830	0·4332	1·0834	1·0833	0·4333	1·0833
5	0·5	1·1327	0·5664	1·1333	1·1332	0·5666	1·1331
6	0·6	1·1966	0·7180	1·1974	1·1972	0·7183	1·1972
7	0·7	1·2769	0·8938	1·2778	1·2776	0·8943	1·2776
8	0·8	1·3761	1·1009	1·3774	1·3771	1·1017	1·3771
9	0·9	1·4970	1·3481	1·4996	(1·4993)	(1·3494)	
				1·4997	1·4993	1·3494	1·4993
10	1·0	1·6470	1·6470	1·6491	(1·6487)	1·6487	
				1·6492	1·6488	1·6488	1·6487
11	1·1	1·8201	2·0120	1·8318	(1·8313)	(2·0144)	
				1·8320	1·8314	2·0145	1·8313
12	1·2	2·0517	2·4620	2·0552	(2·0545)	(2·4654)	
				2·0554	2·0547	2·4656	2·0544
13	1·3	2·3245	3·0218	2·3291	(2·3282)	(3·0267)	
				2·3293	(2·3283)	(3·0268)	
					2·3284	3·0269	2·3280
14	1·4	2·6601	3·7241	2·6660	(2·6648)	(3·7307)	
				2·6663	2·6651	3·7311	2·6645
15	1·5	3·0746	4·6119	3·0823	(3·0808)	(4·6212)	
				3·0827	3·0811	4·6216	3·0802

The cumulative error up to this stage has amounted only to about 0·03%. The extra line in the calculation for $n = 13$ has been inserted because the check value obtained from formula (6) differed slightly from the final interpolated value. The check value was taken as probably the more accurate since it is subject to less rounding-off error, though actually it was less accurate.

A curious feature of this computation is that the first values obtained in column 6 are nearer to the correct values than are the final values. This is not generally the case, however: it is merely a peculiarity of this particular differential equation. In any case we should expect our later results to be too large, since $\dfrac{h^4}{15}\, y_n^{\mathrm{IV}}$ continually increases. In fact, since $y^{\mathrm{IV}} = (x^4 + 6x^2 + 3)y$, $\dfrac{h^4}{15}\, y_{10}^{\mathrm{IV}}$

$\simeq 0{\cdot}0001$; so that errors at least as large as those actually occurring should be anticipated.

A particularly valuable feature of this method is the check on the accuracy of the arithmetic. It shares with Milne's Method II a continuous check against palpable errors in that the values in columns 3 and 5 differ by relatively little, and the amount of the difference varies only slowly. Formula (6), however, provides an overall check that is not available in Milne's method.

There are, of course, much more accurate numerical methods of solving differential equations than that given here; but much more labour is required to obtain the greater accuracy, and some of these, e.g. the Runge-Kutta methods, suffer from the serious drawback that no check is available on the accuracy of the computation.

Although the idea of linear interpolation between a "predicted" and a "corrected" value is an obvious one once it has been pointed out, no use seems to have been made of it hitherto. A method (actually due to W. E. Milne himself) given on pages 134–138 of Milne's Numerical Calculus is an illustration of this. In this method the predictor has an error term $\frac{28}{90} h^5 y^{\mathrm{V}}$ and the corrector an error term $-\frac{1}{90} h^5 y^{\mathrm{V}}$. Milne denotes by D the difference between the values given by the predictor and corrector, and by E_1 and E_2 the errors in the predicted and corrected values, so that $E_2 = D/29$ approximately. So long as $D/29$ is not significant, the corrected value is assumed to be correct. Milne goes on to say "If the error $E_2 = D/29$ proves to be larger than desired accuracy permits, it is necessary to shorten the interval h". In fact, all that is necessary is to interpolate linearly in the ratio 28 : 1 between the predicted and corrected values; and, if this were done throughout, then either the computation could be carried out to a greater number of decimal places or the value of h could be increased.

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2818. On Note 2471

Miss H. P. Hudson says that if a cubic surface R touches a quadric Q at every point of a generator g of Q, then 2 of its 27 lines coincide with g. This is rather misleading, because 2 lines of a cubic surface cannot coincide: either the surface has 27 lines, or it has less, because it has one or more double points.

It may be shown without difficulty that R has a double point (at a point of g), so R has only 21 lines, one of which is g.

E. J. F. PRIMROSE

19. Morley's trisector theorem.

If the 6 interior trisectors of the angles of a triangle ABC are drawn and the pairs of trisectors nearest to each side meet at L, M, N, then LMN is an equilateral triangle.

The direct proofs of this theorem, which prove that LMN is equilateral *after* drawing all the trisectors are fairly complicated. (For one of the simplest, due to W. E. Philip, see Roger A. Johnson *Modern Geometry*, p. 253.)

It is, however, comparatively easy to work backwards from the answer, and show that when the angles of $\triangle ABC$ are given, it is always possible to build up the required configuration, *starting from an equilateral triangle*, since it is obviously only the shape and not the size of the figure which is important. (For a valid proof on these lines, see *Eureka*, no. 16, p. 6.)

To illustrate the dangers of this type of proof, however, consider the following argument.

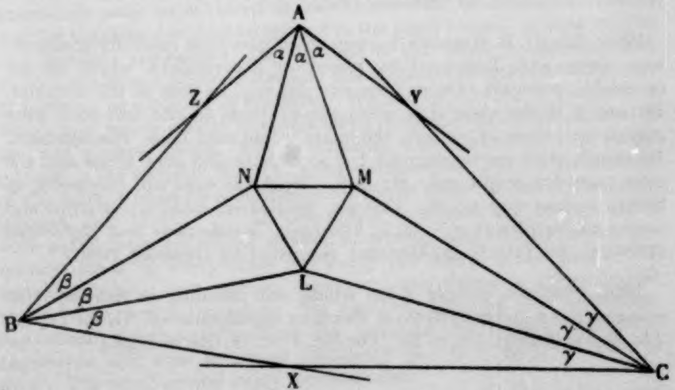
The angles of $\triangle ABC$ are given; for convenience, write $\hat{A} = 3\alpha$, etc. Take an equilateral triangle LMN, and outside this, construct triangles AMN, BNL, CLM, which have the following base angles, θ being arbitrary, but fixed before the construction begins.

$$\angle AMN = \tfrac{1}{3}\pi + \gamma + \theta, \qquad \angle ANM = \tfrac{1}{3}\pi + \beta - \theta,$$

$$\angle BNL = \tfrac{1}{3}\pi + \alpha + \theta, \qquad \angle BLN = \tfrac{1}{3}\pi + \gamma - \theta,$$

$$\angle CLM = \tfrac{1}{3}\pi + \beta + \theta, \qquad \angle CML = \tfrac{1}{3}\pi + \alpha - \theta.$$

Then, since $\alpha + \beta + \gamma = \tfrac{1}{3}\pi$,



$$\angle MAN = \pi - (\tfrac{1}{3}\pi + \beta - \theta) - (\tfrac{1}{3}\pi + \gamma + \theta) = \alpha,$$

and

$$\angle BLC = 2\pi - \tfrac{1}{3}\pi - (\tfrac{1}{3}\pi + \beta + \theta) - (\tfrac{1}{3}\pi + \gamma - \theta) = \tfrac{2}{3}\pi + \alpha.$$

Construct, as shown, $\angle MAY = \angle NAZ = \alpha$, etc., the pairs of lines meeting at X, Y, Z. Then

$$\angle BXC = 2\pi - (\tfrac{2}{3}\pi + \alpha) - \beta - \gamma = \pi.$$

Thus, if the diagram is drawn accurately, B, X, C are collinear, and similarly, so are C, Y, A and A, Z, B.

Hence, whatever angle θ is selected, a $\triangle ABC$ of prescribed angles $(3\alpha, 3\beta, 3\gamma)$ will be obtained, whose angle trisectors meet at the vertices of an equilateral triangle in the required way.

But, obviously, if the construction had been started with $\triangle ABC$, the angle θ would have been determinate and *not* arbitrary.

C. E. Rees

1922. One lecturer was accustomed to show by experiment that a body would fall down a tube in the shape of a cycloid faster than down a tube of another shape, corresponding to the same vertical height; but in order to assist nature he was wont to *grease* the ball surreptitiously which travelled on the cycloid. Again, another lecturer was accustomed to illustrate a mechanical principle known by the name of *virtual velocity;* a certain weight ought to remain immoveable, though not absolutely fixed; in this case a *nail* applied to the weight, unknown to the spectators, prevented any casualty in the experiment. — I. Todhunter, *The Conflict of Studies* (1873), p. 18. [Per G. N. Watson; suggested by Gleaning 1902].

1923. His [H. E. Roscoe's] courses of lectures were carefully arranged, were clear, and illustrated by plenty of experiments, which almost invariably came off. Once, when one did not, he rose to the occasion. He had told the class that when the contents of one test-tube were poured into those of another, the latter would turn blue. His assistant, Heywood, tried the experiment but something had gone amiss and the tube turned a bright red. He said, "Hold the tube up, Heywood, so that the class may see it. You see, gentlemen, what a beautiful and very peculiar blue it is."—J. J. Thomson, *Recollections and Reflections* (1936), p. 28. [Per G. N. Watson; suggested by Gleaning 1902.]

1924. That the Temple of the winds, still standing at Athens, is an octagon, not a dodecahedron, is therefore significant.—E. G. R. Taylor, *The Haven-finding Art*, p. 20. [Per Mr. J. C. W. De la Bere.]

The Teaching of Mathematics. Issued by the Incorporated Association of Assistant Masters in Secondary Schools. Pp. ix, 231. 15s. 1957. (Cambridge University Press)

Operation Technology is under way. The key subject is mathematics and the difficulties and failures in its learning and teaching have become matters of national interest. There is a sense of urgency in the situation and the changed climate of opinion is favourable to educational advance. "Let no one ignorant of Geometry enter here", is the familiar inscription at the entrance to Plato's Academy. Today's 'traditional' attitude is changing. Mathematical ignorance can no longer be paraded as a social grace nor countenanced with complacence. The teaching of mathematics must be considered afresh. This is precisely what the writers of this report set out to do but with a difference. The first consideration is the point of view of the pupil. Mathematics must be related to his needs and contribute to his educational development. This is the correct approach. The subject is then attractive and more and better mathematicians result. The report is the work of a committee of eighteen members and many correspondents representing both the I.A.A.M. and the Mathematical Association. It is an important publication and its appearance is timely and welcome.

Many and various are the factors which govern the teaching of mathematics in any particular type of school or institution. The report deals only with Grammar School mathematics but the authors consider their subject in a broad setting and range widely over the relevant educational field. Topics discussed in the opening chapter include the nature of mathematics, mathematics and mental training, the emotional factor in learning and incentives, and basic teaching theory is deduced. Readers are reminded of the inner and outer aspects of mathematics as described by the late Sir Percy Nunn in his book, 'The Teaching of Algebra including Trigonometry', for the nature of mathematics is seen to present an obstacle to its successful teaching. How can immediate ends be achieved *in association* with ultimate ends? How can the teaching material be related to the pupil's world of ideas to give it immediate relevancy? The attitude of the pupil is acknowledged to be a significant factor in learning and it should be taken into account in teaching as a matter of necessity. We find (p. 11, 12), "The motives which lead young people to specialize in mathematics at the university are not always simple, but presumably one motive is a powerful interest in the subject: a real love for mathematics. The immediate aim of the teacher must be to arouse in his pupils some of this enthusiasm, interest and love . . .. Unfortunately not even knowledge allied with enthusiasm will wholly succeed in encouraging a liking for mathematics unless at every stage we study the needs of the pupil." A correct teaching approach is necessary, one which is psychological rather than logical in which the problem precedes the technique; one which significantly relates the work to the pupil's immediate experience and aims at giving confidence (p. 36) 'even if the pace is slow'. The pupil should enjoy mathematics.

These theoretical considerations are further pursued in subsequent

chapters and their practical implications for teaching worked out. Aims, syllabuses, classroom and homework routine, departmental organisation, sixth form work, library and practical work are carefully discussed and teaching hints are given in abundance. The treatment brims over with idealism and enthusiasm allied with realism and practical common-sense. Under the heading, 'Use of blackboard', for instance, we find, "A teacher should not develop habits in his blackboard work which he intends to discourage his class from adopting in their own written work"' How often do these words have to be repeated to students on teaching practice! Again, "Respect for the metric system should not carry us so far that we teach our children units of area and volume which are unfamiliar on the continent." "The teacher must be the main 'text-book'." "There need be no conflict between disciplined effort and mere pleasure." "Somewhat exceptional qualities are needed in the specialist mathematician if he is to adapt his aims and values to the curious world of mathematics of his pupils. He will find himself compelled to re-interpret his mathematical knowledge."

Teaching syllabuses are distinguished from examination syllabuses and in the third chapter the former are worked out in some detail for A streams to G.C.E. O-level. The keynote is 'mathematics, one subject'. The course specially aims at instilling healthy attitudes to the subject. The first two years are judged to be of particular importance in this connection and are given separate treatment. Syllabuses for the final three years are given in tabular form for convenience, under the five headings, Arithmetic, Algebra, Geometry, Trigonometry, and Graphical Work. A sixth column, History of Mathematics, might well have been added to provide further integrating material and to link with a later section on the subject. The syllabuses are analysed for unifying ideas and principles.

It is recommended in the report on Secondary Education of the Advisory Council on Education in Scotland, issued in 1947, that sizeable Secondary Schools should be equipped with one or more mathematical laboratories. This idea is taken up with some force and developed at length. The committee prefers to call them Mathematics Rooms as more accurately describing what it has in mind and convincing reasons are advanced for their establishment in Secondary Modern and Grammar Schools. Suggestions as to their size, lay-out, equipment, and use are offered in elementary detail despite the fact that very little information was forthcoming from schools. Practical work has probably been found more relevant and useful in Modern than in Grammar Schools and teachers in the former are likely to be ahead in experience. Although this chapter ranks high in importance in the mind of the committee, other chapters also are valuable in suggesting lines of consolidation and advance: 'Some General Teaching Topics' which deals with such items as degrees of accuracy, logarithms, geometry, visual aids, correlation with other subjects, surveying, meteorology, Calculus in the Fifth, and the history and philosophy of mathematics : 'Sixth Form Work', a chapter which raises many questions and reveals the lack of cooperation between schools and universities on the question of

examinations: 'The Mathematical Library', now recognized to be 'indispensable to a broad mathematical education': and 'The Teaching of Statistics'.

Methods of teaching mathematics in the Primary School are briefly described in an early chapter entitled Pre-Secondary School Mathematics and a minimum syllabus is suggested and outlined. There are thoughtful comments on the teaching and on the effects of the 11+ examination which are based on observation of the work of Grammar School entrants and which merit serious consideration. The chapter is primarily concerned with mathematical objectives, the reader being referred to the appropriate Mathematical Association report for a more comprehensive treatment. The main object of the course is stated to be to give facility and confidence in the fundamental operations and processes (p. 18). By contrast, the first fundamental aim for Grammar School children is given as 'the inculcation of a feeling, almost of love, for mathematics' and this is stressed and elaborated in detail. This applies with at least equal force in the Primary School where children are even more sensitive and impressionable and explicit reference to the fact would have been worthwhile. In this respect, the report is written as if children meet mathematics for the first time in the Grammar School. This may be a touch of realism. Nevertheless, the trend in the Primary School in recent years has been away from the attitude which regards arithmetic as little more than a tool subject to be taught as an end in itself, towards one in which arithmetic is seen and approached as the beginning of mathematics. This seems to be a growing point in the teaching of the subject and considerable benefit could accrue from a closer liaison between Primary and Grammar School teachers.

A reprint of the Jeffery Report (1944) concludes the report and an index is provided.

To attempt to collect and summarise available experience on the teaching of mathematics 'still in a state of uneasy transition from the traditional forms which prevailed up to the beginning of the present century', which is the stated purpose of the report, can have been no easy task. The result is a contribution of high order to the literature of the subject. Its influence will be widespread and it should prove helpful in consolidating and stabilising the present position and in preparing the way for further advance. Some will think it conservative when viewed, for example, in the light of the deficiencies in attitudes of intending Primary School teachers noted in the A.T.C.D.E. report, 'The Supply of Mathematics and Science Teachers'. Will teachers find it possible in practice to distinguish between a teaching and an examination syllabus for less able pupils in view of the 'pressure of conditions' (p. 40)? Yet as a guide and companion to present day thought and practice in the teaching of mathematics in the Grammar School, the report sets its own standard. It carries conviction and is free of pedantry. It is authoritative but seems to put the reader in the position of colleague and collaborator. Attractive in style and presentation it will be read widely and with pleasure.

W. Flemming

A New Certificate Arithmetic. (Vol. II) By J. Hislop. Pp. 235. 9s. 1957. (Methuen)

The first volume of this Arithmetic Course showed a refreshing up-to-date treatment of the subject; the present book continues in the same way and has much to commend it. All the usual topics for a G.C.E. course are included (although there is no discussion of graphical methods). The most attractive feature is the section on what could be called the Arithmetic of Citizenship—rates, local government finance, income tax and the budget, insurance of all kinds, foreign exchange, stocks and shares. It is rare to find a text-book in which these matters, so important to the embryo citizen, are treated so fully and with such effect. The actual data used by the organizations concerned are reproduced—the Budget for 1956–7, the estimated expenditure of a typical local council (a copy of an actual rates Demand Form might have increased the effectiveness of the work even more, as pupils would have observed the fact that rates are stated to the tenth of a penny and should wish to know why such accuracy is necessary). A "Notice of Coding" form for Income Tax is given and should provide much material for discussion and obviate the need for subsequent visits to the Inspector of Taxes for explanations. The different forms of Insurance policies are discussed and actual tables used in the large number of illustrative examples.

The chapters on Areas and Volumes are well done with many different types of examples. Facts are stated without proofs (e.g. the volume of a pyramid and the "s-formula" for the area of a triangle), a procedure justified at this stage. The opportunity could have been taken of mentioning the history of the "s-formula" and of Archimedes' proof of the area of a sphere. The formula for the volume of a frustrum of a cone is derived in Algebraic form—rather difficult for the unaided student but a useful piece of knowledge.

The author uses a peculiar method of deriving the simple formula for the areas of parallelogram, triangle, rhombus etc. For example, in obtaining the fact that the area of a rhombus is one-half the product of the diagonals he *states* the geometrical theorem that the diagonals bisect each other at right angles and then sets out the conclusions in the form of a proof—"given", "prove", "proof." All that is surely necessary is to derive the formula by a simple piece of Algebra. The "proof" given is not a proof in the Geometrical sense.

The chapter on logarithms is last in the book—since logarithms are so useful and present little difficulty to an average pupil, they could have been dealt with much earlier (even before a full discussion of the theory of indices). Logarithms should not be regarded as a tool for the expert only but one which is easy to master and use from the earliest moment.

It is important that pupils realise the degree of accuracy that a question will warrant and the author on occasions does not help him. The cost of carpeting and staining a floor (page 20) is not required to the nearest farthing, nor is compound interest (page 178).

This is a very well produced and stimulating book which, with

its companion volume, should be examined by schools searching for an Arithmetic text-book which will be of value to their pupils not for examinations only.

E. H. Copsey

General School Mathematics. Vol. III. By C. G. Paradine and C. Thomas. Pp. xvii, 271, 18. 8s. 6d.; 9s. 6d. with answers. 1957. (The English Universities Press)

The third volume sets out to provide adequate revision of earlier work with sufficient new material in arithmetic, algebra, geometry, and trigonometry for the third year of a five year G.C.E. 'O' level course.

Three chapters are devoted to arithmetic, dealing with compound interest, mensuration of the cone and sphere, with a section on radian measure, and logarithms; five chapters to algebra, dealing with fractions, the remainder and factor theorems, problems leading to quadratic equations, transformation of formulae, and quadratic, cubic, and hyperbolic graphs; five chapters to geometry, covering loci and chord properties of circles, angle properties of circles, areas and the theorem of Pythagoras with the extension theorems, and tangent properties of circles; and two chapters to trigonometry, covering the tangent and co-tangent, sine and cosine and relationships, and three dimensional problems. The final chapter contains 15 short mixed test papers, followed by 9 test papers in arithmetic, 9 in algebra, and 11 in geometry and trigonometry. A set of mathematical tables concludes the book.

Each chapter is a collection of related topics, treated by the traditional methods, each topic being illustrated by worked examples followed by an exercise of similar problems, except for Ex 3(a) which is unrelated to the topic on page 38. Of all the chapters, that on logarithms is, in the opinion of the reviewer, the least satisfactory, the discussion on the determination of the characteristic being somewhat confusing. It would have been better to have started the section by expressing numbers in standard form and thus developing the rule for the determination of all characteristics by this method. This should have been used in finding both logarithms and anti-logarithms (a rule on page 137 would have been useful).

On page 63 rules are given for solving problems leading to quadratic equations, but no mention is made of the fundamental rule that we should express each part of the resulting equation in the same unit.

There are a number of small errors, both in the printing and in the working. On page 95, $2{\cdot}7x$ should read $2.7x$; on page 149 £21·75 = £21·15s is bad; on page 206, fig. 125, 70° not 70′; on page 21 line 3, R is omitted from fig. 14; on page 23 $\times$ is omitted from fig. 17; on page 30 line 3, O should be Q; on page 88 the reason for the congruence of Δ's ABH, ACH is omitted; on page 155 the multiplication is correct to 3 figures; and on page 58 Ex 4(d) question 4, small c not C. Other small points include: Ex 3(c) the answer to question 25 is $(n - 3m)^2(n + 3m)$; on page 140, $1{\cdot}4609 \div 3 = 0{\cdot}4870$; on page 145, $\bar{1}{\cdot}9540$ should be encouraged rather than $\bar{1}{\cdot}954$; and on page

140, $\pi = 3{\cdot}14$ in a logarithm exercise on the circle is hardly justifiable.

Though clearly printed with adequate explanations and clear diagrams, judged from the standard of the exercises, particularly the geometry, this volume could be recommended for the abler mathematicians only. For the less able pupil there are insufficient questions of the drill type. The value of a main school textbook depends upon its abundance or otherwise of examples to be worked by the pupil, and consequently the amount of explanation should be reduced to its bare minimum. This volume does err on the side of too few questions.

E. C. Witcombe

Calculus. By Jack R. Britton. Pp. xiv, 584. 52s. 1957. (Rinehart, New York)

This book provides a leisurely introduction to the differential and integral calculus. The first 15 chapters (about three-fifths of the whole) deal with the elementary theory of differentiation and integration of functions of one variable. There is little here which is unusual in such introductions, save for the treatment of the exponential function (which compares unfavourably with that in the Association's report on the teaching of calculus). The author takes for granted the existence of a continuous function a^x (which he calls *the* exponential function) possessing the usual properties of a power. The logarithmic function $\log_a x$ is then defined as the inverse of a^x. The number e is introduced as the limit of $(1 + h)^{1/h}$ as $h \to 0$ (the existence of this limit, too, is taken for granted), and this limit is then used to calculate the derivative of $\log_a x$. Finally the derivatives of e^x and a^x are obtained from the result concerning the differentiation of an inverse function.

The order of the material in this part of the book could sometimes be improved, e.g. the result of the mean value theorem is used on p. 231 (as "geometrically evident") although the actual theorem is not stated until p. 298. A section on improper integrals in the middle of a chapter on the mean value theorem also seems very much out of place. There is a serious error in the proof of L'Hospital's theorem (p. 301), where the ordinary mean value theorem is used in place of Cauchy's form. There is also a remarkable account of centrifugal force on p. 250 which appears to repudiate Newton's laws of motion.

The remaining chapters of the book deal with infinite series, approximate integration, partial differentiation, multiple integrals, and elementary differential equations. The account of infinite series is a good treatment at an elementary level, though marred by a nonsensical definition of an infinite series as "the collection of symbols $u_1 + u_2 + u_3 + \ldots + u_n + \ldots$". The discussion of partial differentiation can only add confusion to a subject in which confusion already abounds. The formula for the total differential of a function $f(x, y)$ of 2 independent variables appears only under the hypothesis that f_x and f_y are continuous. In all theorems subsequent to this, however, the formula is used under the condition of differentiability, although the concept of

differentiability for functions of several variables is not defined anywhere in the book!

The book concludes with a list of integrals, a list of infinite series, 12 pages of drawings of various curves, and 26 pages of numerical tables. There are copious worked examples and exercises, answers being given for the odd-numbered exercises. There is also a comprehensive index.

In his preface the author claims that it is his purpose to present the material "as simply as possible, but with due regard for the modern requirements of rigour . . . procedures and theorems are made plausible; but plausibility arguments are not allowed to pass as proofs". In certain instances the author achieves these aims by quoting fundamental and difficult theorems without any indication that they are anything other than trivial. For example, the result that a continuous function attains all values between its bounds is dismissed in the middle of a proof (p. 112) with only a prefatory "by virtue of the continuity", while uniform continuity is used in an even more off-hand manner (p. 273). It is questionable whether the author's aims are praiseworthy in a first course on the calculus. In any case, it seems to the reviewer that it is preferable in a first course to be completely unrigorous than to achieve rigour by such means as these.

T. M. Flett

Algebra. By C. O. Tuckey and W. Armistead. Pp. 402. 18s. 1958. (Longmans and Green)

This book covers the algebra syllabuses with the associated analysis, for the various A levels in pure mathematics; and contains the additional topics required for the main mathematics papers of science open scholarships.

The book abounds in good worked examples, exercises, and test papers, and the treatment shows the hand of the practised teacher. In particular one was glad to see power series being so called and being fully equated to their functions.

If the book may seem to some to be over-full (with e.g. 60 pages of mainly O-level revision at the start) this is a good fault: even allowing for this and for overlap with one's calculus text-book the work is good value. One wonders however whether a separate volume containing chapters XI–XVI might not have a market of its own, for post-A-level work with the scientist.

A. J. Moakes

Basic Physics. By J. Meetham. Pp. 134, 21s. 1958. (Pergamon Press)

The content of this book is described as A-level theoretical physics shorn of its terrestrial bias, and it is meant for the reader who is

educated in science but who is not a specialist. "The aim is to describe the physical world . . . as simply and consecutively as possible."

In a small compass a considerable number of facts of varying significance are included, with related formulae and with diagrams which are assumed to be self-explanatory. Possibly one has been spoiled by modern standards of exposition, but this treatment was disappointingly arid—and the price would buy 5 months' copies of a lively weekly publication aimed at a similar public.

A. J. MOAKES

Elementary Mechanics. By QUADLING and RAMSAY. Vol. I. Pp. 272. 14s. 1957. (Bell, London)

This book is the first of three volumes intended to provide a course in Mechanics up to University work, and introduces the basic ideas of the subject in covering the ground needed for G.C.E. at O level. The authors have succeeded in introducing principles which cannot be dealt with rigorously at this stage, by means of a practical discussion of well chosen examples and the resulting book work is clear and easy to understand. The importance of a clear diagram in the solution of Mechanics problems is stressed. The second chapter of the book is devoted to this topic and is followed by a set of examples on drawing diagrams showing all the forces acting in a wide variety of situations, which, if carefully discussed, would be of great value to the beginner.

There is no division into Statics and Dynamics, the two developing side by side quite naturally, so avoiding any feeling on the part of the beginner that here are two separate subjects each to be tackled by its own exclusive methods.

Absolute units are used with the equation of motion in the form $P = mf$, thus avoiding much of the confusion over units which is so common later amongst the weaker boys who have used gravitational units initially.

An interesting feature of the book is the use made of vectors, much more space being devoted to this topic than is customary in elementary books. The operations of addition, subtraction, and multiplication by a scalar are introduced, accompanied by worked examples and sets of examples to make the notation clear and familiar. Reference to vectors is made subsequently whenever possible, as for example, in the chapter on Momentum when a vector triangle is used to find an impulse vector as a difference between two momentum vectors. This means, presumably, that in the next two volumes vector methods will be featured prominently, and so will help to bridge the gap which certainly exists at present between School and University Mechanics.

The examples in the book are plentiful and well chosen to arouse the interest of the Scholars who will be working them, and there are four sets of Miscellaneous Exercises which are excellent for Revision Work.

F. E. CHETTLE

Solution of Problems in Strength of Materials. By S. A. URRY. 2nd ed. Pp. 406. 20s. 1957. (Pitman)

Each chapter of this book begins with a number of worked examples in which the theory of the subject is developed and by means of which the method of application of the theory to specific problems is demonstrated. These worked examples are followed by a set of questions by means of which the student may test the knowledge he has acquired; numerical answers to all of these questions are provided at the end of each question.

The book is divided into 15 chapters and the topics dealt with are, in this order, simple stress and strain, elasticity, shearing force and bending moment, longitudinal stresses in beams, torsion of circular shafts, deflection of beams, propped beams, cantilevers, encastre beams, struts, complex stresses, principal strains and the elastic constants, thin shells, thick cylindrical shells, strain energy, springs, shearing stresses in beams, mechanical properties, and theories of failure. Then follow two sets of test papers and two appendices, the first of which deals with the analysis of experimental results and the second with moments of area. No previous knowledge of the subject is assumed but the standard reached can be judged from the fact that the majority of the questions are taken from past papers set at the examinations of the Professional Engineering Institutions and for the Engineering degree of the University of London. Numerical illustrations are given at all stages and one very welcome feature is the insistence on the need for stating the units in which both final and intermediate results are measured.

One gains the impression that the book is the outcome of several years' successful teaching of the subject: it may well be recommended (a) to students who wish to supplement their lecture notes with a very useful collection of illustrative examples and (b) to young teachers who might welcome the help to be obtained from such a book in presenting the subject of their students. The book is well printed on good paper and is cheap at the price.

F. T. CHAFFER

Observation and Interpretation. A Symposium of Philosophers and Physicists. Pp. 218. 40s. 1957. (Butterworths, and Academic Press, New York)

This volume of lectures on the philosophy of physics is remarkable both for the range of the speakers and for the vivid reproduction of discussions. Amongst the most interesting is a sketch of a projected theory by David Bohm, of Israel, which aims at a higher degree of determinism than is possible in quantum theory. Bohm's theory met with a good deal of hostile criticism from other speakers.

R. L. G.

Recursive Number Theory. By R. L. Goodstein. Pp. ix, 190. 36s. 1957. Studies in logic and the foundations of mathematics. (North-Holland, Amsterdam)

Recursive number theory is the study of the natural numbers in which only recursive functions and relations may be defined. This class of functions and relations is frequently further narrowed to the primitive recursive functions and relations. Roughly speaking, these are the functions (and relations) whose values can be effectively calculated by a step-by-step procedure, the value for $n + 1$, say, depending on the value for n. A simple example is the definition (by primitive recursion) of the sum function:

$$a + 0 = a$$
$$a + (b + 1) = (a + b) + 1.$$

The notion of adding 1 is taken as fundamental. Thus, for any natural numbers a, b, the sum $a + b$ can be evaluated in $b + 1$ steps, starting with $a + 0 = a$, and ending with $a + b = (a + (b - 1)) + 1$. $a + b$ is a primitive recursive function of two arguments. One may build up more complicated functions by substitution, e.g. $a + (b + c)$, and by further recursions, e.g.

$$a \,.\, 0 = 0$$
$$a \,.\, (b + 1) = a \,.\, b + b.$$

We are allowed to use the sum function in the latter definition since it was previously defined as a primitive recursive function. The constant functions and the identity functions of any number of arguments, i.e. $f(x_1, \ldots, x_n) = x_i$, $1 \leq i \leq n$, are also admitted as primitive recursive.

In a fundamental paper (Skrifter Norske Videnskaps-akademi, 1923) Th. Skolem showed that much of elementary number theory could be obtained in primitive recursive arithmetic. In particular, he showed that unique factorization is provable. The theory was further advanced by K. Gödel in his famous paper on undecidable propositions. Professor Goodstein is among the important contributors to the theory subsequent to that. The present work includes expositions of results in the general theory of primitive recursive functions as well as results belonging to arithmetic as such. A wider class of functions (the multiple recursive functions) is introduced, but only the primitive recursive functions enter essentially.

The Introduction is a skilful and illuminating discussion of the nature of the natural numbers. Just as the king of chess is defined implicitly by the rules of chess, in particular, the specification of the king's moves, the natural numbers are defined implicitly by the rules of their theory. One never plays chess with the king of chess, but only with an object assigned the role of king; and one does arithmetic by operating with numerals, which play the role of numbers.

Chapter I introduces the notion of definition by recursion, and several important primitive recursive functions are defined. The reader should note that "the general notion of a recursive function" as given on p. 24, is different from, and properly included in, the notion of general recursive function. The author mentions the distinction in the notes to

the bibliography. General recursive functions do not enter in the present work.

Chapter II introduces the equation calculus, a formal system of recursive arithmetic, its assertions are of the form $F = G$, where F and G are (primitive or multiple) recursive functions. The techniques of proof in the calculus are fully and carefully illustrated. An equation $F = G$ is called verifiable if substitution of numerals for the variables in the equation always reduces F and G to the same numeral. The chapter ends with a proof that only verifiable equations are provable in the system. For this proof, the reader must understand that the reduction of F and G to numerals does not involve the rules of inference of the formal system. The technique of this reduction is explained in Chapter I.

In Chapter III, the propositional calculus is constructed within the equation calculus. The procedure is that discovered independently by Professor Goodstein and by H. B. Curry. An equation $a = b$, where a and b are numerals, is called a proposition. Defining a primitive recursive function $f(x, y)$ such that $f(x, y) = 1$ if $x = y$, and $f(x, y) = 0$ if $x \neq y$, we call the proposition $f(a, b) = 0$ the negation of $a = b$. Similarly, the other propositional operations are defined. The bounded quantifiers are defined, as well as a counting operator, which, for a given function $f(x)$ and a given number n, has value $k =$ the number of solutions of the equation $f(x) = 0$, $0 \leq x \leq n$.

There are some minor errors in this chapter. In particular, necessary restrictions on the variables are not stated for some of the proof schemata, e.g. schemata 3·951, 3·952, and 3·96. The proofs of these results should make it clear what the restrictions are to be.

The notions of quotient, remainder, prime number, and common divisor are introduced in Chapter IV. It is shown that some of the standard results of elementary number theory, such as the unique factorization theorem, are derivable in the equation calculus.

Chapter V concerns formalizations of primitive recursive arithmetic. The axioms are primitive recursive function definitions, and (in the final form of the system) two particular equations. The rules express the usual properties of equality, and, instead of the usual non-elementary induction rule, we have a rule that from equations $f(0) = 0$ and $f(n) = f(n + 1)$, we may derive $f(n) = 0$. This is an extremely elegant form of recursive arithmetic.

It is shown in Chapter VI that certain kinds of definitions by recursion, although not primitive recursions, can be reduced to primitive recursions. Among these are course-of-values recursions, in which the value $f(n + 1)$ depends not just on $f(n)$, but on some or all of the values $f(0), f(1), \ldots, f(n)$.

Let $x \dot{-} y$ be $x - y$ if $x \geq y$ and 0 otherwise. Let $Rt(x)$ be the greatest integer y such that $y^2 \leq x$. These two functions are primitive recursive. The principal result of Chapter VII is as follows: Assume that $x + y$, $x \,.\, y$, $x \dot{-} y$, and $Rt(x)$ are given functions, along with the successor function $f(x) = x + 1$, the zero function, and the identity functions (the statement of this result at the top of p. 139 fails to

mention these latter functions). Then all primitive recursive functions can be defined by substitution and application of the schema

$$f(0) = 0$$
$$f(x + 1) = g(f(x))$$

where $g(y)$ is a previously defined primitive recursive function of one variable. Functions of more than one variable are defined with the aid of primitive recursive pairing functions $J(x, y)$, $U(x)$, and $V(x)$ which have the property that $U(J(x, y)) = x$, and $V(J(x, y)) = y$.

Chapter VIII, the last chapter of the book, uses results from many of the preceding sections in order to derive a form of Gödel's incompleteness theorem for primitive recursive arithmetic. Recursive arithmetic furnishes a rather obvious context for Gödel's argument, so that it is possible to present a full treatment of the theorem in just ten pages. Finally, Skolem's non-standard model for arithmetic is constructed.

Professor Goodstein gives a very readable and complete exposition throughout. The book would be suitable for study by undergraduates. Its value is increased by inclusion of generous lists of examples to be worked by the reader. The examples not only test the reader's grasp of the techniques, but their results are referred to in the text. Complete solutions to all the examples are given at the end of the book.

There are a number of misprints, mostly of a trivial nature, such as mis-numberings of formulas and interchangings of capital and lower-case letters. The reviewer also has one serious reservation about the book: the author does not carefully distinguish between formal proofs and meta-mathematical arguments. The consistency proof of Chapter II suffers on this account, and so does the incompleteness proof of Chapter VIII. Both are meta-mathematical results, but nowhere is the reader told of this. The incompleteness theorem is subtle; the reader can only be confused by sections like 8.89, where it is by no means clear how the word "holds" is to be interpreted in its several occurrences.

The importance of the subject, the selection of topics, and the great skill at exposition generally displayed make this a welcome and valuable work.

BRUCE LERCHER

Introductory Formal Logic of Mathematics. By P. H. NIDDITCH. Pp. 188. 12s. 6d. 1957. (University Tutorial Press).

This book is the first of a projected series on "logical mathematics"; in the preface Dr. Nidditch writes, "The book initiates a radically new departure in mathematics, since it gives for the first time logically valid proofs of mathematical theorems." This statement ignores the work of Russell and Whitehead, Hilbert, Gödel, and other mathematical logicians; and it implies the false assertion that there is a wide gap between the best modern standards of rigour, and logical validity. Nevertheless, were the project properly carried out it would undoubtedly be of some interest, and might even produce a surprise or two.

But, alas, when it comes to dealing with *formal systems* Dr. Nidditch

falls far below the standards which he seeks to impose on *mathematics*; all is obscurity and confusion. This is sad, because the book—intended partly as a textbook of mathematical logic—is one of the few to be based on Gentzen's admirable "system of natural deduction." Despite a reference to the work of Kleene, Dr. Nidditch appears ignorant of those metamathematical conventions and techniques (developed during the last thirty years) which make it possible to express statements *about* formal systems succinctly, lucidly, and without ambiguity. In this book there are no metamathematical notations at all, the same variable may be used for both objects and expressions, and there is frequent confusion between use and mention. (To say "if a formula is a set" is as ungrammatical as to say "if a noun is a table.")

Even by less sophisticated standards, the presentation is intolerably lax: definitions contain terms which are neither self-explanatory nor elsewhere defined; derived rules are quoted but not proved; and on p. 82 "in cases like Sxx: . $_y$by" occurs as a clause of definition. These are pedantic criticisms, but here pedantry is in order; what would Dr. Nidditch say of an author who used "functions like sin x" as a definition? However more than mere pedantry is involved, for—because of the looseness of their formulation—at least three of the rules of inference (I16, V25, VIII26) are not valid and are liable to lead to contradictions. Until these matters have been cleared up *the majority of the proofs given are logically suspect.* The book can only serve as a model of how not to do it.

R. O. Gandy

Colloque de Topologie Algébrique. (G. Thone. Liège) 1956

The C.B.R.M. (Centre Belge de Recherches Mathématiques) organized a colloquium in 1950 devoted to the topology of fibre-spaces. Thus the Colloquium held in Louvain in June, 1956, may be regarded as the second colloquium held under the auspices of the C.B.R.M. since the war to be devoted to algebraic topology—proud record!

The colloquium was an occasion of great value to those concerned and the present volume should extend that value beyond the domain of its participants.

P. J. Hilton

Physical Properties of Crystals. By J. F. Nye. Pp. xv, 322. 50s. 1957. (Oxford: Clarendon Press)

No one can understand crystal physics unless he is reasonably familiar with tensor and vector methods. For these alone give a rational basis for incorporating the symmetry properties which distinguish the regular arrangement of the atoms in a crystal from their irregular arrangement in an amorphous solid. In this book Dr. Nye gives a

theoretical treatment of the macroscopic (i.e. large-scale) properties of crystals in a form that should be readily understandable to any normal second year honours physics student.

The author first develops the necessary tensor and vector analysis (50 pages). No previous familiarity with these ideas is assumed, and the treatment is exceptionally clear and well set out. Next (150 pages) he discusses the equilibrium properties of a crystal, such as stress, strain, electric and magnetic susceptibilities, and piezoelectricity, including one chapter on the alternative matrix formulation and another on the thermodynamics of the equilibrium properties of crystals. Piezoelectricity introduces us to tensors of rank 3, and elasticity to tensors of rank 4. This leads into the third section (41 pages) on transport properties, including thermal and electrical conductivity separately and then their superposition as in thermoelectricity. This could be called thermodynamics in distinction to the previous thermostatics. The field is now considerably more difficult, and the central theorem (Onsager's Principle and Casimir's objection to it) has to be stated in a rather *ex cathedra* manner because its proper development belongs to statistical mechanics. The point at issue, which is excellently brought out, is whether, if the presence of an antisymmetrical part in the thermal and electrical conductivity tensors does not affect any observational properties, we may fairly put it equal to zero and so neglect it. Finally (40 pages) there is an account of crystal optics, which deals with double refraction and optical activity.

The outstanding feature of the book is its attention to good teaching practice. The whole account is unusually clear for a topic that so easily becomes obscure. This is almost the only book in this field of which I know, that could be put in the hands of a typical student with the expectation that he would be able to read it and understand it without help. The illustrations are a model of what such things should be, and the mathematics is clear and well-arranged.

But there are certain penalties that have to be exacted. The most serious of these is the complete neglect of all microscopic (i.e. atomic) effects, on which ultimately all symmetry properties rest. The word "atom," for example, does not even appear in the index. It does seem a small pity that not even a simple explanation of the major physical phenomena could have been given in these terms. It would then have been possible to include some account of the Cauchy relations; as it is, these are not even referred to by name. The book would also have been improved if a little more space could have been given to ferromagnets, ferroelectrics, semi-conductors, and hysteresis effects. But if this suggestion is made, the author has an almost cast-iron reply: "to include any of these in other than an unsatisfactory phenomenological way would have required that very atomic outlook which has been rejected as unsuitable for an introductory account of this type."

Dr. Nye is to be warmly congratulated on an excellent book, which it is a pleasure to handle and to read.

C. A. Coulson

Irrational Numbers. By IVAN NIVEN. Pp. 164. 1956. Carus Mathematical Monographs No. 11)

It is claimed that the monographs of this series are comprehensible, in particular, to "the wide circle of thoughtful people who, having a moderate acquaintance with elementary mathematics†, wish to extend their knowledge without prolonged and critical study of the mathematical journals and treatises." Judged by this claim the book must be counted a failure. But, because the author has made a determined and persistent attempt (except perhaps in the last two chapters) to present his subject in the simplest possible way, he has succeeded in writing a book, which can, in the main, be understood by those with a moderate knowledge of elementary mathematics, provided they are prepared to take a certain amount on trust, and to make a prolonged and critical study of its pages (unfortunately the proviso cuts out most circle-squarers). Further, and perhaps more important, the book can be read with enjoyment, but not without effort, by those with a more mature knowledge of mathematics.

Throughout the standard of exposition is high; there seems to be a very close correlation between the difficulty of the different sections and the inherent difficulty of the topics discussed. But, I personally, would have found the book much easier to read, if the author had attempted to give some sort of motivation for the various tricks used in the proofs. Even if, on closer examination, such motivations do not seem to be completely convincing, they are often of great help during the first reading of the work. In my opinion, there are two things, which should not have been omitted:— the first an exposition of Cantor's diagonal argument proof that the real numbers are not countable (the author instead works along measure theory lines, basing his proof on a claim, that it is in a sense intuitively clear, that the interval from 0 to 1 cannot be covered by a set of intervals of arbitrarily small total length); the second a proof of the transcendence of π which is as simple as possible (the author proves the general Lindemann theorem, but does not spare the space necessary to give a simpler special discussion of π).

C. A. ROGERS

Niels Henrik Abel. By OYSTEIN ORE. Pp. 277. 45s. 1957. (Minnesota University Press. Oxford University Press).

When a young man dies at the age of 26 years, the acknowledged equal of the greatest mathematicians of his time, his story cannot fail to be moving in the extreme. How did it come about that the son of a well to do family, his father a one time member of the Norwegian parliament, a mathematical genius who won world wide recognition by the time he was 23 years of age, died in want, never having held more than a temporary appointment? Where did the blame lie? What share of the responsibility must be born by Gauss, Cauchy and the King of Sweden? This is the mystery story which the book tells, and tells with affection, understanding and well authenticated detail.

R. L. G.

† Interpreted, in this review, as meaning the body of mathematics usually taught at school.

"Der Begriff des Raumes in der Geometrie":—Bericht von der Riemann—Tagung des Forschungsinstituts für Mathematik. Edited by J. Naas and K. Schröder. Pp. 317. D.M. 38. 1957. (Akademie-Verlag, Berlin)

This report contains 28 papers read at a Conference held in Berlin, October 11th–16th, 1954 to celebrate the centenary of Riemann's famous inaugural dissertation "Über die Hypothesen welche der Geometrie zugrunde liegen." The authors include A. D. Alexandrow, G. Alexits, W. Blaschke, E. Cech, M. Fiedler, H. Freudenthal, S. Golab, M. Haimovici, O. Haupt, E. Hölder, L. K. Hua, L. Infeld, G. Klapka, K. Kuratowski, G. Lampariello, P. Libois, Z. Nadenik, N. Obreschkoff, A. Papapetrou, H. Rund, K. Schröder, F. Severi, K. Strubecker, B. Sz.-Nagy, J. Tits, O. Varge, G. Vranceanu, and F. Vycichio.

The papers are all concerned with some aspect of the concept of space in geometry, and the relation between the papers and Riemann's dissertation is briefly discussed in an introduction by J. Naas. The frontispiece is a photograph of Riemann, while at the end of the book are facsimiles of two interesting letters, one in Riemann's handwriting to L. Kronecker and a second letter to Kronecker written by Riemann's wife after her husband's death.

T. J. Willmore

Leçons sur les Récurrences et leur Applications. By Paul Montel. Collected and edited by J. Dufresnoy and E. Lefebvre. Pp. 268. 4300 fr. 1957. (Paris, Gauthier-Villars)

A new book by this famous French mathematician attracts obvious interest. In this latest contribution to the well known Collection Borel we have Montel's lecture course on the theory of recurrence to which topic he himself has contributed much. It deals both with sequences $\{u_n\}$ satisfying a recurrence

$$u_{n+p} = f_n[u_n, u_{n+1}, \ldots u_{n+p-1}]$$

and with functions $u(x)$ whose differences satisfy a recurrence

$$\Delta^p u(x) = \varphi[u(x), \Delta u(x), \ldots \Delta^{p-1} u(x), x].$$

Such recurrences are at the basis of many analytical concepts and processes, and Montel's account ranges over a wide field of such applications: Schröder functions, difference equations, continuous fractions, and the theory of iteration and fixed points.

Montel did not write the book himself, but his lecture notes have been edited and presented by two of his pupils. Though the master himself acknowledges, in the preface, the elegance and clarity of their rendering, the reviewer cannot share this view. To him their conversational treatment (so charming in a good Cours d'Analyse) appears both lax and unprecise. Thus, again and again, vital assumptions are mentioned en passant and, without further emphasis, sometimes used and sometimes not. Theorems are rarely formulated and still more rarely exhibited in special print. Thus the reader will look in vain

for a formulation of the theorem of Poincaré after which Chapter V is named. Again, the technique required in proofs and exposition is often heavy [this may be unavoidable], but little effort has been made to use simple notations and, worse still, to stick to them. Frequent misprints have been overlooked and add to the difficulty of reading the book. It is rather a pity that such a valuable and interesting account of an important topic has been much spoiled by a careless rendering.

W. W. ROGOSINSKI

Probability. By M. T. L. BIZLEY. Pp. 229. 20s. 1957. (Cambridge University Press)

The probability examination paper set by the Institute of Actuaries used to consist of tricky combinatorial problems concerning the seating of men and wives at dinner and the like so ably delineated and solved by W. A. Whitworth in his *DCC Exercises on Choice and Chance* of 1895, a book which the reviewer considers for this particular type of probability problem will never be equalled. It is important therefore in considering Mr. Bizley's book, sponsored as it is by the Institute of Actuaries, to note that there is little here which will be of help in calculating "seating" odds. The book is in fact yet another elementary probability text book and it is not particularly well done. Chapter I has an exposition of what the author considers are fundamental principles, Chapter II covers rules for the combination of probabilities, Chapter III the theorems of Waring and Boole, and Chapter IV repeated trials. There follows a little on difference equations and a little on runs of luck, and a final chapter (VII) on the continuous variable. The exposition is clear but somewhat pedestrian. The printing, as always with the Cambridge Press, is excellent. F. N. DAVID

Techniques of Mathematical Analysis. By C. J. TRANTER. Pp. 396. 27s. 6d. 1957. (English Universities Press)

This is a very useful account suitable for the mathematical specialist in his last year at school and the first year student reading for a General Degree. The style is crisp and clear and the worked examples numerous and interesting.

The title is well chosen because, apart from a brief reference to Dedekind sections and an incomplete proof of the existence of a least upper bound (p. 42) the theory of real numbers is not discussed. The following minor unclarities are perhaps worth noting. Abel's proof of the impossibility of solving a quintic by radicals (p. 135) is surely beyond controversy; I cannot see the difficulty mentioned in p. 157 in proving that the complex circular functions are periodic with period 2π; the inequality $-1 \leq \cos\theta + i\sin\theta \leq 1$ on p. 168 needs correction; the example on p. 189 may be solved simply by differentiating the determinant; on p. 210 it is necessary to add the restriction $\phi(x) \neq \alpha$ for x in a neighbourhood of a; on p. 239 the possibility that $\delta x = 0$ has been overlooked and on p. 325 the definition of envelope is unsound.

R. L. GOODSTEIN

Experimental Designs. By W. G. Cochran and G. M. Cox. 2nd ed. Pp. xiv, 611. 82s. 1957. (John Wiley and Sons, New York; Chapman and Hall, London)

The first edition of this book was reviewed in the Feb., 1952, issue of the *Mathematical Gazette* (36 (315) 78–79). In it we noted that the book was "intended as a source of reference . . . , primarily a handbook" and that the "real value of such a handbook can . . . only properly be assessed by the worker who has occasion constantly to refer to it . . . "

The book, with, in particular, the sets of plans of experimental designs, has been much used by such workers, and the new selection should be even more valuable, for it adds, to all those designs originally given, many of the newer types, with indices of other designs now available, from any of which the worker can make his choice. There are, of course, many difficulties that may arise if "home made" designs are used, and it is of great value to have to hand in one volume details of such a variety of approved designs, with notes on their use. The framework of the book is unchanged, and the chapter and section number references are unaltered. To each chapter are now added further references to the literature, and a list called "Additional Reading", though the "Selected Bibliography" at the end of the book is now much reduced—to one page of eighteen references, the majority of them new. At the end, after pagination is closed, we now have five pages of tables of t and of F: these have appeared in others of the *Wiley Publications in Statistics*.

Though there are a few omissions of former sections and paragraphs (and some modifications of some of the plans and of the calculations) the book is now much bigger, as a number of sections have been expanded, and two new chapters (called 6A and 8A) have been added: a lot of chapter 11 is also new. Fuller treatment is now accorded to designs to deal with experiments where the experimental units are classified into two classes (e.g., dead and alive), where different treatments are applied in time sequence to the same subjects (e.g., in feeding experiments on dairy cows, when the whole experiment must be completed in one lactation) and in other cases of sequential experimentation, where five or more factors can be included simultaneously in an experiment of practicable size (e.g., in industry), where we have experiments involving polynomial response surfaces (a topic e.g., dealt with extensively by G.E.P. Box by his method of steepest ascent), and those where we can use "chain block" designs (in which the number of treatments considered exceed the size of block-experiments involving a "linking" method appropriate to cases in the physical sciences). Of special interest to teachers will be the sections dealing with the "staircase" sequential experimentation on lines used in determining "mental age" by Binet testing, and those dealing with multiple comparisons and with balanced incomplete blocks in taste and preference testing, with their applications to the class ranking of pupils.

The new book should thus be of much value to all who wish to use modern designs in their experimentations.

Frank Sandon

Les Algèbres d'Opérateurs dans l'Espace Hilbertien. (Algèbres de von Neumann.) By J. DIXMIER. Pp. 367. 5500 fr. 1957. (Gauthier-Villars, Paris)

The study of algebras of operators in Hilbert space was initiated by von Neumann nearly 30 years ago; it developed from his work on spectral resolution of operators, which in turn arose out of his study of the mathematical foundations of quantum theory. Since that time it has been studied by numerous mathematicians, and has found important applications in the modern theories in which algebra and analysis meet, notably in the theory of representations of topological groups.

The book under review is the first in any language devoted to the study of this theory. That this is so, in spite of the importance of the theory, is in part to be explained by the circumstance that attention has been diverted from the theory of operators in Hilbert space to the more general theory of Banach algebras, which has proved in some respects simpler than the von Neumann theory as well as extremely fruitful in its applications to analysis. Nevertheless, the earlier theory has retained its importance within the framework of the more general theory; and as expositions of the theory of Banach algebras exist in English, M. Dixmier's book fills a definite gap in mathematical literature.

The author has, indeed, concentrated his attention on this gap, and his text avowedly avoids exposition both of the general theory of Banach algebras and of the applications to analysis of the theory of algebras in Hilbert space. The book is divided into three parts, devoted respectively to Global theory, to reduction theory, and to Complements—more specialized topics. The chapter on Global theory commences with the definition of von Neumann algebras, the name given by the author to the weakly closed algebras which contain the adjoint of any operator in them; M. Dixmier defines them in a more algebraic fashion, and proceeds in this first chapter to study in the main the algebraic properties and structure of these algebras. Thus, products and tensor products of algebras, positive linear forms and traces on the algebras, factors—i.e. algebras with only scalars in the centre—are discussed. The third chapter is mainly devoted to studies of the same type as the first but of a more specialized nature—in particular to a deeper study of the theory of factors. The second chapter is concerned with the theory of direct integrals of Hilbert spaces, and with the corresponding direct integrals of measurable fields of algebras; a theory which to some extent generalizes to algebras the spectral resolution theory for operators. This chapter depends strongly on the Bourbaki theory of integration—which in its turn must have been influenced by the type of problem arising in this theory—and involves some deep parts of set theory. Finally, the book has five appendices which give—somewhat briefly—accounts of theories of which use is made in the book; some sections of Hilbert space operator theory, the Gelfand representation theory, and some unpublished results of Bourbaki concerning analytic sets.

The book is well and carefully written. The author has been at

pains to facilitate the reading of it. He has provided useful indices of notations and terminology, as well as a comprehensive bibliography. In addition, what is unusual in books at this level, he has given extensive and interesting exercises which serve to inform the reader about other developments of the theory; usually references to papers involved in the subjects of the exercises are given, and in other cases hints facilitate solution.

The book, however, does need considerable previous knowledge for its reading, since the Bourbaki theories of topology and topological vector spaces, the theory of operators in Hilbert space and the Gelfand representation theory for Banach algebras are presupposed, as well as some aspects of the Bourbaki theory of integration. This makes it a work of specialized appeal; but to those who are interested in its subject it will be invaluable. The production is of the usual high standard of the house of Gauthier-Villars.

J. L. B. COOPER

BRIEF MENTION

An Introduction to Probability Theory and its Applications. By W. FELLER. 2nd Ed. Pp. 461. 86s. 1958. (Chapman & Hall)

The second edition has new chapters on random walks and compound distributions.

Probability, Statistics and Truth. By R. von Mises. 2nd Ed. Pp. 244. 28s. 1957. (Allen & Unwin)

The new English edition has been prepared by the author's widow Hilda Geiringer using a German text revised by the author in 1951.

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